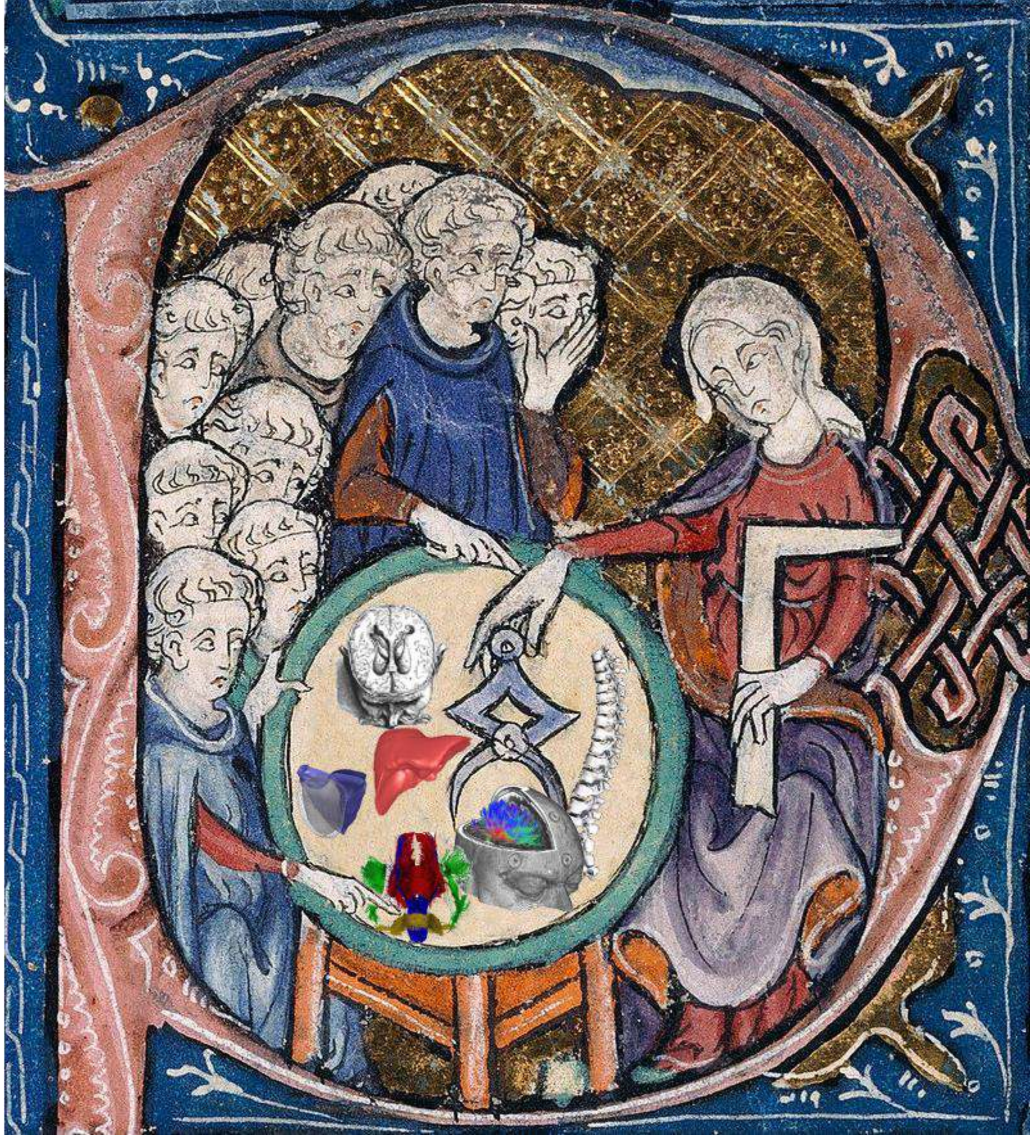
Xavier Pennec

Asclepios team, INRIA Sophia-Antipolis – Mediterranée, France

Alain Trouvé

Ecole Normale Supérieure Paris-Saclay



Minicourse Shape Spaces and Geometric Statistics

TGSI, Luminy 31-08-2017

école normale supérieure paris–saclay

ASCLEPIOS

Inria

Shape Spaces and Geometric Statistics

Shape spaces: quotient or not quotient?

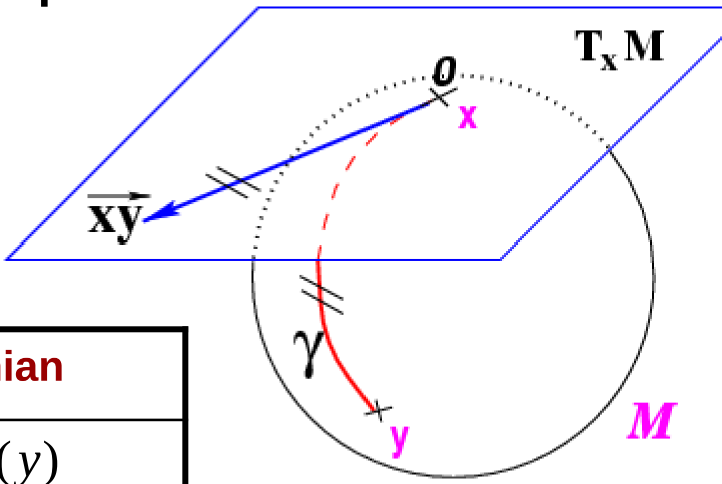
Geometric statistics

- Simple statistics on Riemannian manifolds
- Subspaces for PCA in manifolds
- Perspectives, open problems

Computing in Riemannian Manifolds

Exponential map (Normal coordinate system):

- **$Exp_x(v)$** = geodesic shooting at x parameterized by the initial tangent vector v
- **$Log_x(y)$** = development of the manifold in the tangent space along geodesics
  - Geodesics = straight lines with Euclidean distance
  - Local → global domain: star-shaped, limited by the cut-locus
  - Covers all the manifold if **geodesically complete**

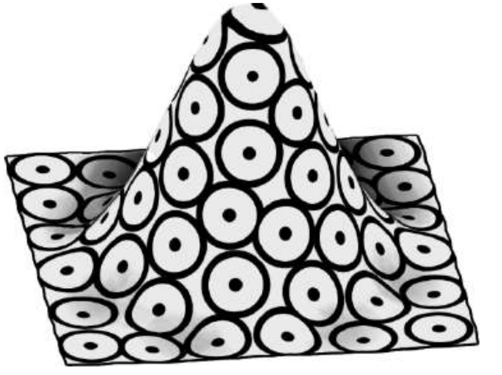


Reformulate algorithms with exp_x and log_x

Vector -> Bi-point (no more equivalence classes)

Operation	Euclidean space	Riemannian
Subtraction	$\overrightarrow{xy} = y - x$	$\overrightarrow{xy} = \log_x(y)$
Addition	$y = x + \overrightarrow{xy}$	$y = \exp_x(\overrightarrow{xy})$
Distance	$\mathrm{dist}(x, y) = \lVert y - x \rVert$	$\mathrm{dist}(x, y) = \lVert \overrightarrow{xy} \rVert_x$
Gradient descent	$x_{t+\varepsilon} = x_t - \varepsilon \nabla C(x_t)$	$x_{t+\varepsilon} = \exp_{x_t}(-\varepsilon \nabla C(x_t))$

Random variable in a Riemannian Manifold



Intrinsic pdf of x

- For every set H

$$P(\mathbf{x} \in H) = \int_H p(y) dM(y)$$

- ~~Lebesgue's measure~~

→ Uniform Riemannian Mesure $dM(y) = \sqrt{\det(G(y))}\, dy$

Expectation of an observable in M

- $\boldsymbol{E}_{\mathbf{x}}[\phi] = \int_M \phi(y) p(y) dM(y)$
- $\phi = dist^2$ (variance) : $\boldsymbol{E}_{\mathbf{x}}[dist(.,y)^2] = \int_M dist(y,z)^2 p(z) dM(z)$
- $\phi = \log(p)$ (information) : $\boldsymbol{E}_{\mathbf{x}}[\log(p)] = \int_M p(y) \log(p(y)) dM(y)$
- ~~$\phi = x$ (mean) : $\boldsymbol{E}_{\mathbf{x}}[\mathbf{x}] = \int_M y\, p(y) dM(y)$~~

Statistical tools: Moments

Frechet mean [1944] minimize the variance

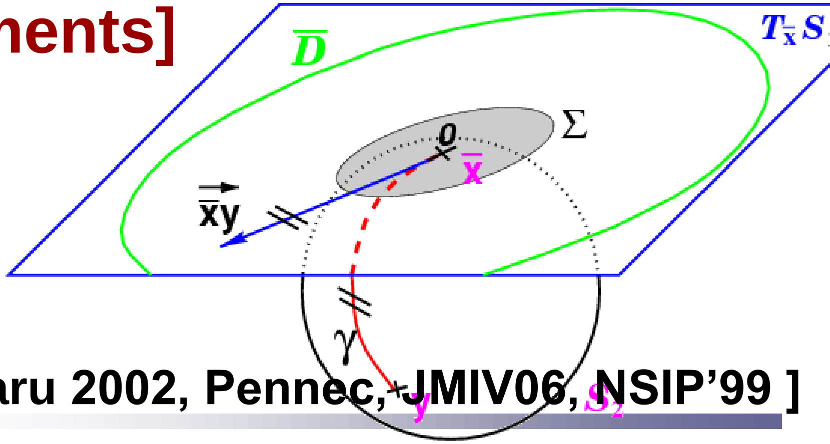
- $\sigma^2(x) = \int_M dist^2(x,z)\, p(z) dM(z)$
- Tensor moments of a random point with density p
  - $\mathfrak{M}_1(x) = \int_M \overrightarrow{xz}\, p(z) dM(z)$ Tangent mean field
  - $\mathfrak{M}_2(x) = \int_M \overrightarrow{xz} \otimes \overrightarrow{xz}\, p(z) dM(z)$ Covariance field

Exponential barycenters are critical pts of variance (P(C) =0)

- $\mathfrak{M}_1(\bar{x}) = \int_M \overrightarrow{\bar{x}z}\, p(z) dM(z) = 0$ (implicit definition of $\bar{x}$)

Covariance [and higher order moments]

- $\mathfrak{M}_2(\bar{x}) = \int_M \overrightarrow{\bar{x}z} \otimes \overrightarrow{\bar{x}z}\, p(z) dM(z)$



[Oller & Corcuera 95, Battacharya & Patrangenaru 2002, Pennec, JMIV06, NSIP'99]

Riemannian center of mass, Fréchet or Karcher mean?

A naming dispute [Karcher 2014]

- For simply connected Riemannian manifolds of non-positive curvature, the minimum of the variance is unique [Cartan 1929]
- Exponential barycenter called THE Riemannian center of mass (under some uniqueness assumptions) [Groove & Karcher 1973-1976]
- Current conventions in geometric statistics:
  [**Fréchet** 1944] set of **global** minima / [**Karcher** 1977] = set of **local** minima
  [Emery 1991, Oller & Corcuera 1995] Exp. barycenters = set of critical points

Existence and uniqueness (extensions to p-means)

[Karcher 77 / Buser & Karcher 1981 / Kendall 90 / Afsari 10 / Le 11]

- Support of distribution in a a regular geodesic ball U(r) with radius $r < r^* = \frac{1}{2}\min(inj(M), \pi/\sqrt{\kappa})$ (κ upper bound on sect. Curv. on M with convention that ${}^1/_{\sqrt{\kappa}} = +\infty \; for \; \kappa \leq 0$)

Empirical mean (finite sample): almost surely unique!

[Arnaudon & Miclo 2013]

Algorithms to compute the mean

Karcher flow (gradient descent)

$$\bar{x}_{t+1} = \exp_{\bar{x}_t}(\epsilon_t \, v_t) \;\; with \;\; v_t = \mathrm{E}(\overrightarrow{\mathrm{y}\mathbf{x}}) = \frac{1}{\mathrm{n}} \sum_i \log_{\bar{x}_t}(x_i)$$

- Usual algorithm with $\epsilon_t = 1$ can diverge on SPD matrices [Bini & Iannazzo, Linear Algebra Appl., 438:4, 2013]
- Convergence for non-negative curvature (p-means) [Afsari, Tron and Vidal, SICON 2013]

Inductive / incremental weighted means

- $\bar{x}_{k+1} = \exp_{\bar{x}_k}\left(\frac{1}{k} \, v_k\right) \;\; with \;\; v_k = \log_{\bar{x}_k}(x_{k+1})$
- On negatively curved spaces [Sturm 2003], BHV centroid [Billera, Holmes, Vogtmann, 2001]
- On non-positive spaces [G. Cheng, J. Ho, H. Salehian, B. C. Vemuri 2016]

Stochastic algorithm

- [Arnaudon & Miclo, Stoch. Processes and App. 124, 2014]

Distributions for parametric tests

Generalization of the Gaussian density:

- Stochastic heat kernel p(x,y,t) [complex time dependency]
- Wrapped Gaussian [Infinite series difficult to compute]
- Maximal entropy knowing the mean and the covariance

$\beta = 0$ for symmetric spaces

$$N(y) = k\ exp(-\beta^t \overrightarrow{\bar{x}\boldsymbol{x}} - \tfrac{1}{2} \overrightarrow{\bar{x}\boldsymbol{x}}^t\ \Gamma\ \overrightarrow{\bar{x}\boldsymbol{x}})$$

$$\boldsymbol{\Gamma} = \boldsymbol{\Sigma}^{(-1)} - \tfrac{1}{3}\mathrm{Ric} + O(\sigma) + \varepsilon(\sigma / r)$$

$$k = (2\pi)^{-n/2}.\det(\boldsymbol{\Sigma})^{-1/2}.\left(1 + O(\sigma^3) + \varepsilon(\sigma / r)\right)$$

Mahalanobis D2 distance / test:

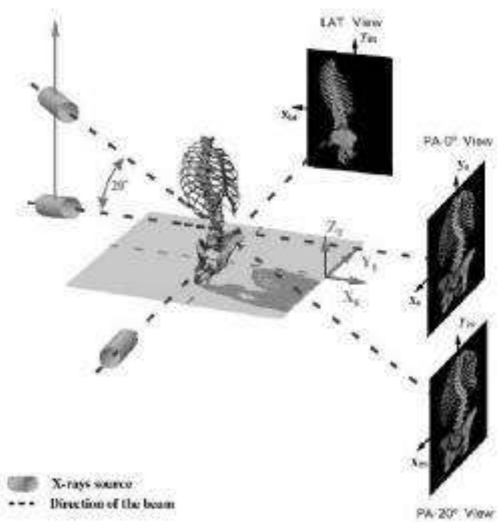
$$\mu_{\mathbf{x}}^2(\mathrm{y}) = \overrightarrow{\bar{\mathrm{x}}\mathrm{y}}^t.\Sigma_{\mathbf{xx}}^{(-1)}.\overrightarrow{\bar{\mathrm{x}}\mathrm{y}}$$

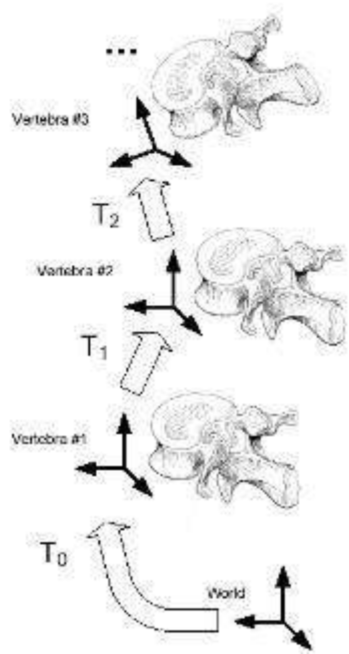
- Any distribution: $\mathrm{E}\left[\mu_{\mathbf{x}}^2(\mathbf{x})\right] = n$
- Gaussian: $\mu_{\mathbf{x}}^2(\mathbf{x}) \propto \chi_n^2 + O(\sigma^3) + \varepsilon(\sigma / r)$

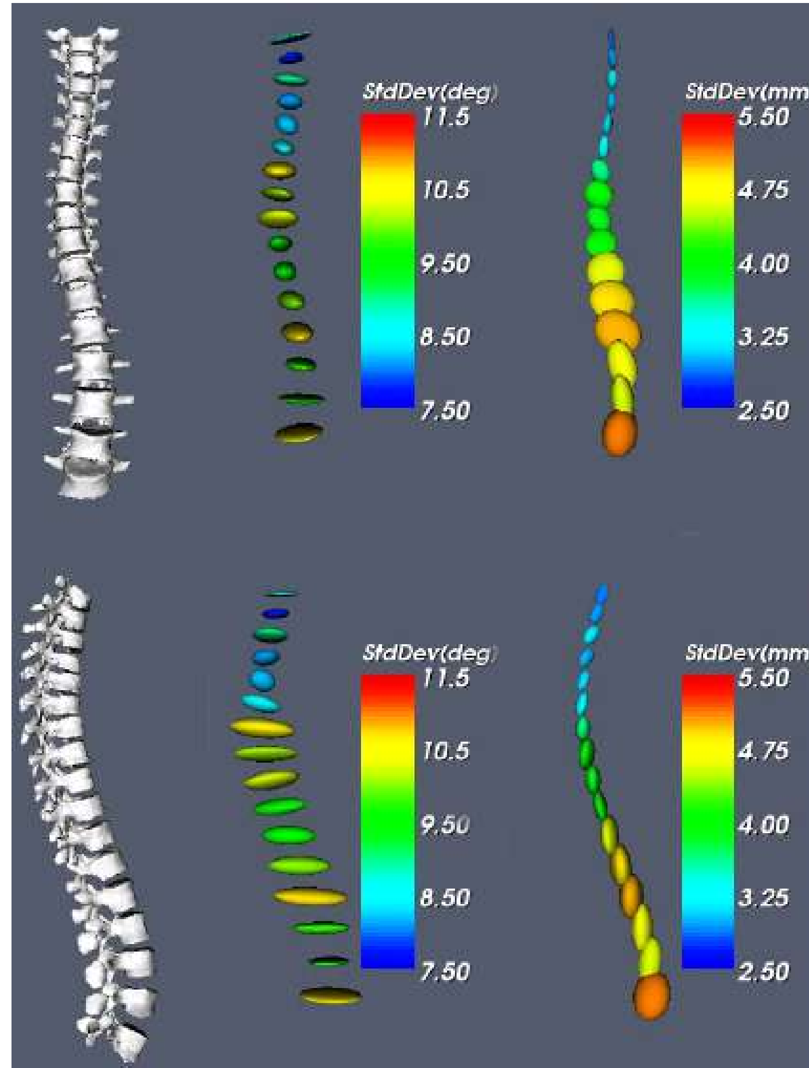
[Pennec, NSIP'99, JMIV 2006]

Statistical Analysis of the Scoliotic Spine

[J. Boisvert et al. ISBI'06, AMDO'06 and IEEE TMI 27(4), 2008]







Database

- 307 Scoliotic patients from the Montreal's Sainte-Justine Hospital.
- 3D Geometry from multi-planar X-rays

Left invariant Mean on $(SO_3 \ltimes R^3)^{16}$

- Main translation variability is axial (growth?)
- Main rot. var. around anterior-posterior axis

Statistical Analysis of the Scoliotic Spine

[J. Boisvert et al. ISBI'06, AMDO'06 and IEEE TMI 27(4), 2008]
AMDO'06 best paper award, Best French-Quebec joint PhD 2009

PCA of the Covariance:
4 first variation modes have clinical meaning

- Mode 1: King's class I or III
- Mode 2: King's class I, II, III
- Mode 3: King's class IV + V
- Mode 4: King's class V (+II)

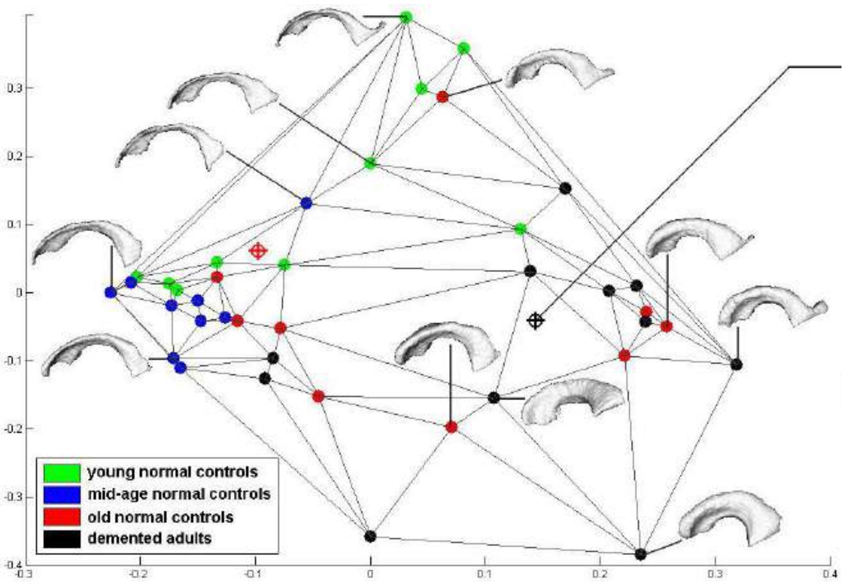
Shape Spaces and Geometric Statistics

Shape spaces: quotient or not quotient?

Geometric statistics

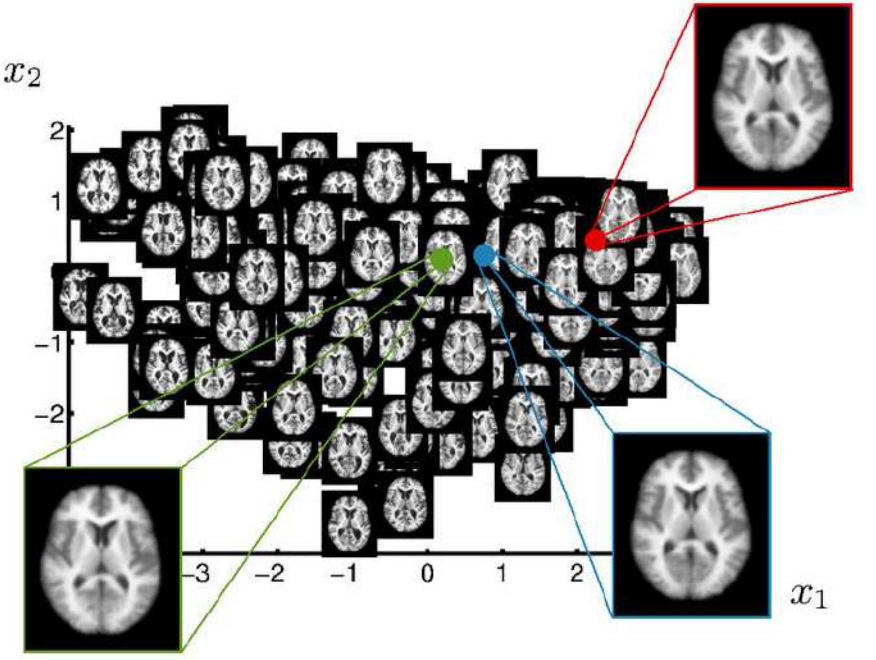
- Simple statistics on Riemannian manifolds
- Subspaces for PCA in manifolds
- Some perspectives and open problems

Low dimensional subspace approximation?



Manifold of cerebral ventricles

Etyngier, Keriven, Segonne 2007.



Manifold of brain images

S. Gerber et al, Medical Image analysis, 2009.

- Manifold dimension reduction
- When embedding structure is already manifold (e.g. Riemannian): Not manifold learning (LLE, Isomap,…) but **submanifold learning**

Tangent PCA

Maximize the squared distance to the mean (explained variance)

- Algorithm
  - Find the Karcher mean $\bar{x}$ minimizing $\sigma^2(x) = \sum_i dist^2(x, x_i)$
  - Unfold data on tangent space at the mean
  - Diagonalize covariance $\Sigma(x) \propto \sum_i \overrightarrow{\bar{x}x_i}\, \overrightarrow{\bar{x}x_i}^t$

- Generative model:
  - Gaussian (large variance) in the horizontal subspace
  - Gaussian (small variance) in the vertical space

- Find the subspace of $T_x M$ that best explains the variance

Principal Geodesic / Geodesic Principal Component Analysis

Minimize the squared Riemannian distance to a low dimensional subspace (unexplained variance)

- PGA (Fletcher et al., 2004, Sommer 2014):

  space generated by geodesics rays originating from Karcher mean:

  $GS(x, w_1, \ldots w_k) = \{\exp_x(\sum_i \alpha_i w_i) \;\; for\ \alpha \in R^k \}$

- Geodesic PCA (GPCA, Huckeman et al., 2010):

  space generated by principal geodesics that cross at one point (principal mean, may be different from Karcher mean)

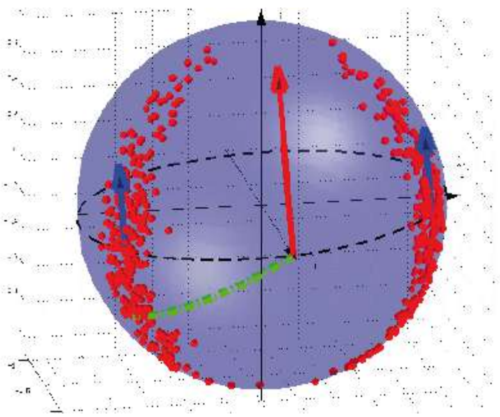
- Generative model:
  - Unknown (uniform ?) distribution within the subspace
  - Gaussian distribution in the vertical space
  - Beware: GS have to be restricted to be well posed

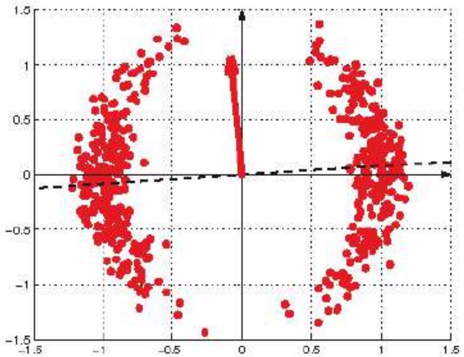
All different models in curved spaces (no Pythagore thm)

Problems of tPCA / PGA

Analysis is done relative to on point

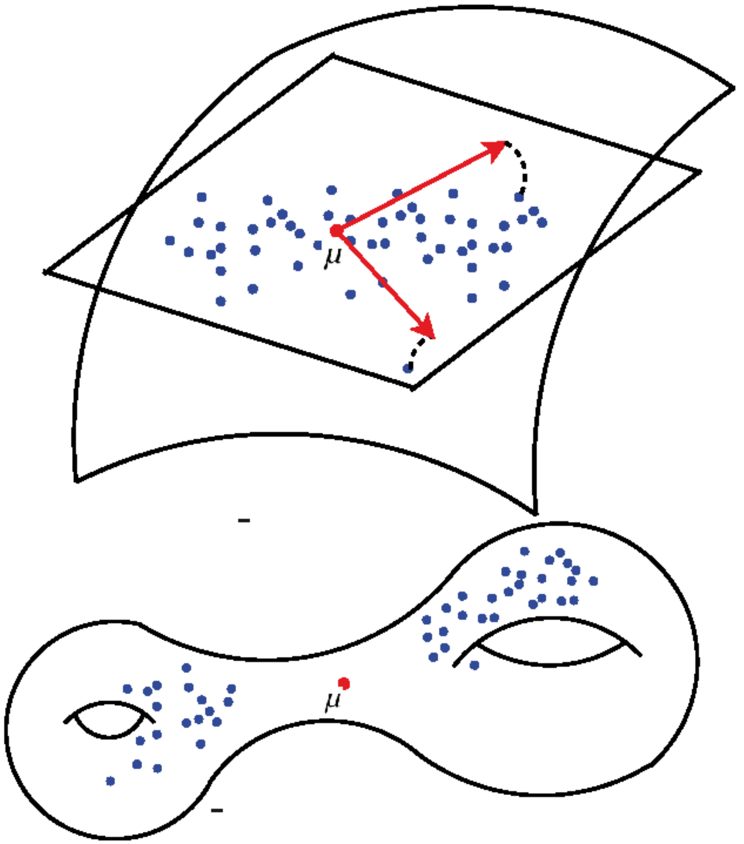
- What if this point is a poor description of the data?
  - Multimodal distributions
  - Uniform distribution on subspaces
  - Large variance w.r.t curvature





Bimodal distribution on S2

Courtesy of S. Sommer



Courtesy of S. Sommer

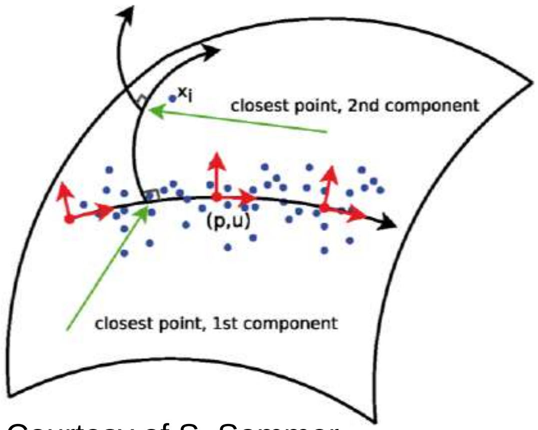
Patching the Problems of tPCA / PGA

Improve the flexibity of the geodesics

- 1D regression with higher order splines **[Vialard, Singh, Niethammer]**
  - **Control of dimensionality for n-D Polynomials on manifolds?**

Iterated Frame Bundle Development
[HCA, Sommer GSI 2013]

- Iterated construction of subspaces
- Parallel transport in frame bundle
  - **Intrinsic asymmetry between components**



Courtesy of S. Sommer

Nested "algebraic" subspaces

- Principal nested spheres **[Jung, Dryden, Marron 2012]**
- Quotient of Lie group action **[Huckemann, Hotz, Munk, 2010]**
  - **No general semi-direct product space structure in general Riemannian manifolds**

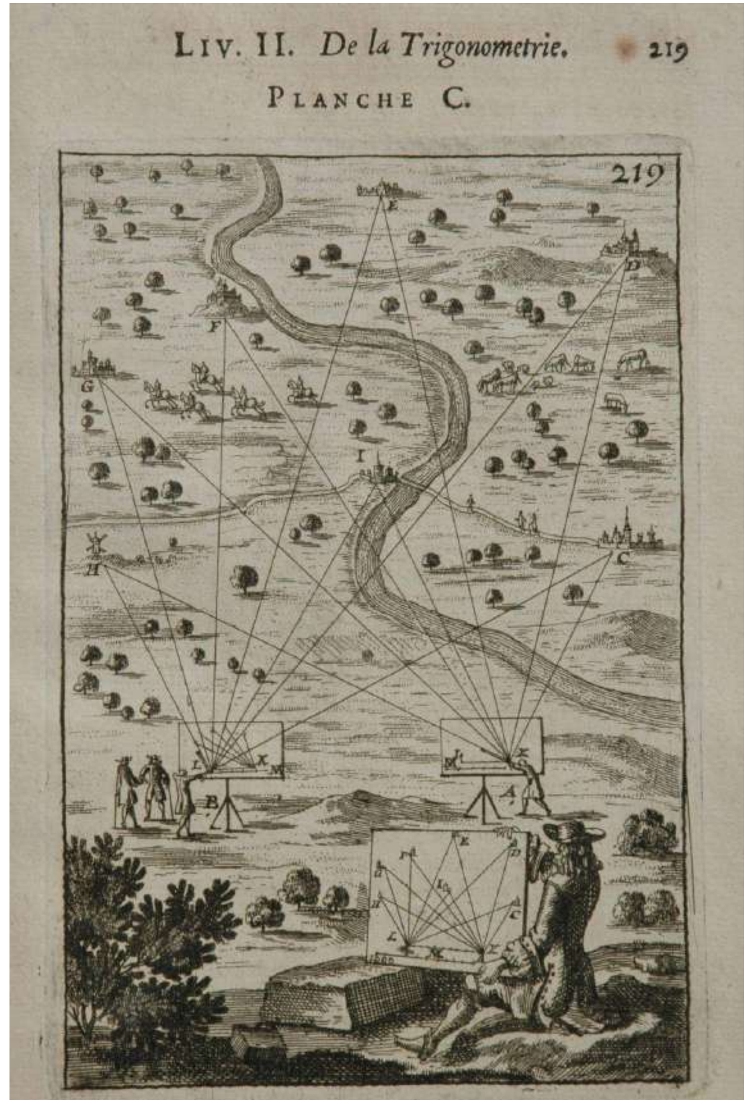
Affine span in Euclidean spaces

Affine span of (k+1) points: weighted barycentric equation

$$\mathrm{Aff}(x_0, x_1, \ldots x_k) = \{x = \textstyle\sum_i \lambda_i x_i \ \ with \ \sum_i \lambda_i = 1\}$$
$$= \{x \in R^n \ \ s.t \ \textstyle\sum_i \lambda_i (x_i - x) = 0, \lambda \in P_k^*\}$$

Key ideas:

- ~~Look at data points from the mean (mean has to be unique)~~
- Look at several reference points from any point of the manifold subspace
- locus of weighted mean



A. Manesson-Mallet. La géométrie Pratique, 1702

Barycentric subspaces and Affine span in Riemannian manifolds

Fréchet / Karcher barycentric subspaces (KBS / FBS)

- Normalized weighted variance: $\sigma^2(x,\lambda) = \sum \lambda_i dist^2(x, x_i) / \sum \lambda_i$
- Set of absolute / local minima of the weighted variance
- Works in stratified spaces (may go accross different strata)
  - Non-negative weights: Locus of Fréchet Mean [Weyenberg, Nye]

Exponential barycentric subspace and affine span

- Weighted exponential barycenters: $\mathfrak{M}_1(x,\lambda) = \sum_i \lambda_i \overrightarrow{x x_i} = 0$
- EBS$(x_0, \ldots x_k) = \{ x \in M^*(x_0, \ldots x_k) \, | \mathfrak{M}_1(x,\lambda) = 0\}$
- Affine span = closure of EBS in M $\quad Aff(x_0, \ldots x_k) = \overline{EBS(x_0, \ldots x_k)}$

Questions

- KBS/FBS defined by minimization: existance and uniqueness?
- Local structure: local manifold? dimension? stratification?
- Relationship between KBS $\subset$ FBS, EBS and affine span?

Analysis of Barycentric Subspaces

Assumptions:

- Restrict to the punctured manifold $M^*(x_0, \dots x_k) = M \,/ \cup C(x_i)$
- Affinely independent points:
  $\{\overrightarrow{x_i x_j}\}_{0 \leq i \neq j \leq k}$ exist and are linearly indep. for all I

Local well posedness for the barycentric simplex:

- EBS / KBS are well defined in a neighborhood of reference points
- For reference points in a sufficiently small ball and positive weights: unique Frechet = Karcher = Exp Barycenter in that ball: smooth graph of a k-dim function [proof using Buser & Karcher 81]

SVD characterization of EBS: $\mathfrak{M}_1(\mathrm{x}, \lambda) = Z(x)\lambda = 0$

- SVD: $Z(x) = [\overrightarrow{x x_0}, \dots \overrightarrow{x x_k}] = U(x) S(x) V^t(x)$
  - $EBS(x_0, \dots x_k) =$ Zero level-set of l>0 singular values of Z(x)
  - Stratification on the number of vanishing singular values

Analysis of Barycentric Subspaces

Exp. barycenters are critical points of w-variance on M*

- $\nabla\sigma^2(x,\lambda) = -2\mathfrak{M}_1(x,\lambda) = 0$ $\quad\quad$ $\boldsymbol{KBS} \cap \boldsymbol{M}^* \subset \boldsymbol{EBS}$

Caractérisation of local minima: Hessian (if non degenerate)

$$H(x,\lambda) = -2\sum_i \lambda_i D_x \log_x(x_i) = \mathbf{Id} - \frac{\mathbf{1}}{\mathbf{3}}\mathbf{Ric}\left(\mathfrak{M}_2(\mathbf{x},\boldsymbol{\lambda})\right) + \text{HOT}$$

Regular and positive pts (non-degenerated critical points)

- $\boldsymbol{EBS^{Reg}(x_0, \ldots x_k) = \{\, x \in Aff(x_0, \ldots x_k), s.t.\ H(x, \lambda^*(x)) \neq 0\}}$
- $\boldsymbol{EBS^{+}(x_0, \ldots x_k) = \{\, x \in Aff(x_0, \ldots x_k), s.t.\ H(x, \lambda^*(x))\ Pos.def.\}}$

Theorem: $\boldsymbol{KBS} = \boldsymbol{EBS}^+$ plus potentially some degenerate points of the affine span and some points of the cut locus of the reference points.

X.P. Barycentric Subspace Analysis on Manifolds [arXiv:1607.02833]

KBS / FBS with 3 points on the sphere

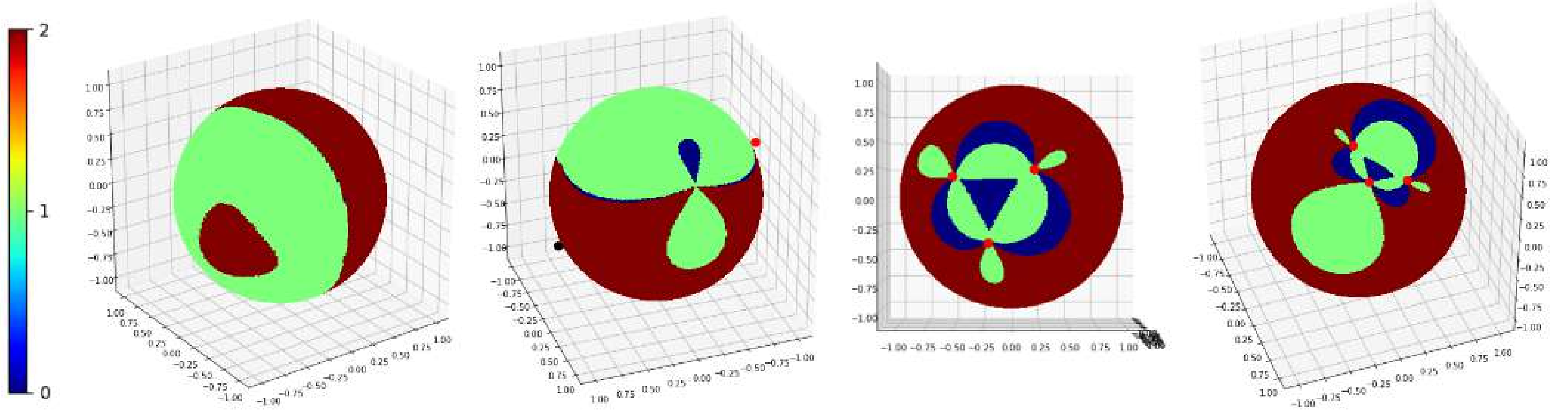
EBS: great subspheres spanned by reference points (mod cut loci)

$\text{EBS}(x_0, \dots x_k) = Span(X) \cap S_n \setminus Cut(X) \qquad Aff(x_0, \dots x_k) = Span(X) \cap S_n$

Index of the Hessian of weighted variance:

$$\mathrm{H}(\mathrm{x},\lambda) = \sum \lambda_i \theta_i \cot(\theta_i)\,(\mathrm{Id} - \mathrm{xx}^t) + \sum (1 - \lambda_i \theta_i \cot(\theta_i))\, \overrightarrow{xx_i}\, \overrightarrow{xx_i}^t$$

- Complex algebric geometry problem **[Buss & Fillmore, ACM TG 2001]**
- In practice positive & negative eigenvalues: KBS/FBS is a strict and incomplete subset of EBS (**less interesting than affine span**)



KBS / FBS with 3 points on the hyperbolic space

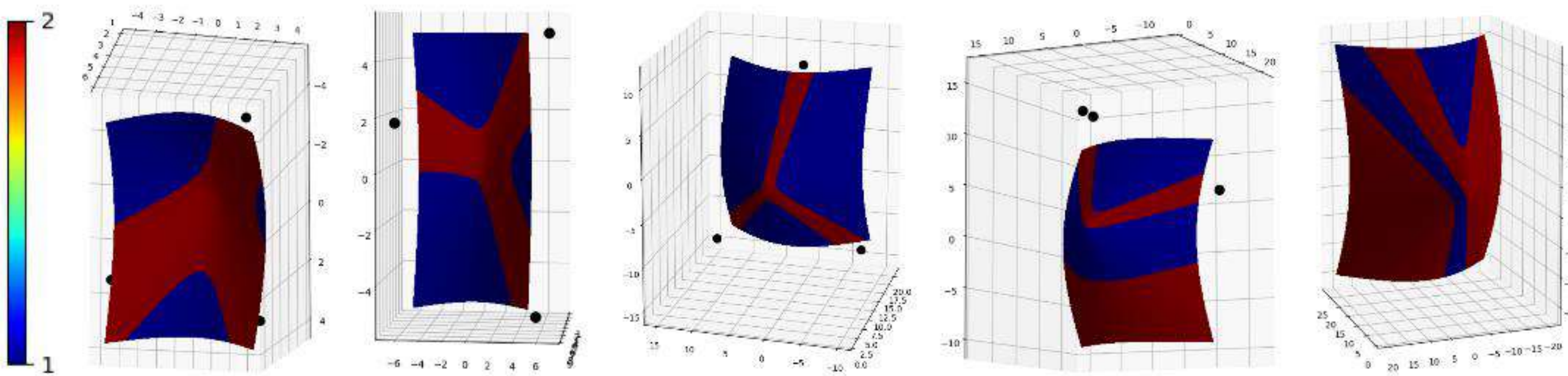
EBS = Affine span: great sub-hyperboloids spanned by reference points

$$\mathrm{EBS}(x_0, \dots x_k) = Aff(x_0, \dots x_k) = Span(X) \cap H_n$$

Index of the Hessian of weighted variance:

$$\mathrm{H}(\mathrm{x},\lambda) = \sum \lambda_i \theta_i \coth(J + J\mathrm{x}\mathrm{x}^{\mathrm{t}} J^t) + \sum (1 - \lambda_i \coth(\theta_i)) J\, \overrightarrow{xx_i}\, \overrightarrow{xx_i}^t J^t$$

- Complex algebric geometry problem
- Better than for spheres, but still disconnected components



Shape Spaces and Geometric Statistics

Shape spaces: quotient or not quotient?

Geometric statistics

- Simple statistics on Riemannian manifolds
- Subspaces for PCA in manifolds
- Perspectives, open problems

Barycentric subspaces

Generalization to α–barycentric subspaces (median, mode)?

- $\sigma^{\alpha}(x,\lambda) = \frac{1}{\alpha} \sum \lambda_i dist^{\alpha}(x, x_i) / \sum \lambda_i$
- Well… critical points of $\sigma^{\alpha}(x,\lambda)$ are also critical points of $\sigma^2(x,\lambda')$ with $\lambda'_i = \lambda_i \, dist^{\alpha-2}(x, x_i)$ (i.e. the affine span)

Limit of affine span for collapsing points

- 1st order: EBS converges to [restricted] Geodesic Subspace
- Conjecture: generalization to non-local higher order (k-n)-jets
  - Principal nested spheres [Jung, Dryden, Marron 2012]
  - Quadratic, cubic splines [Vialard, Singh, Niethammer]
  - Quotient of Lie group action [Huckemann, Hotz, Munk, 2010]

A canonical generalization of affine subspaces in Manifolds?

- Variational formulation? Link with minimal surfaces?

The natural object for PCA: Flags of subspaces in manifolds

Subspace approximations with variable dimension

- Optimal unexplained variance → non nested subspaces
- Nested forward / backward procedures → not optimal
- Optimize first, decide dimension later → Nestedness required
  [Principal nested relations: Damon, Marron, JMIV 2014]

Flags of affine spans in manifolds

- $FL(x_0 \prec x_1 \prec \cdots \prec x_k)$ sequence of nested subspaces $Aff(x_0, \ldots x_i)$

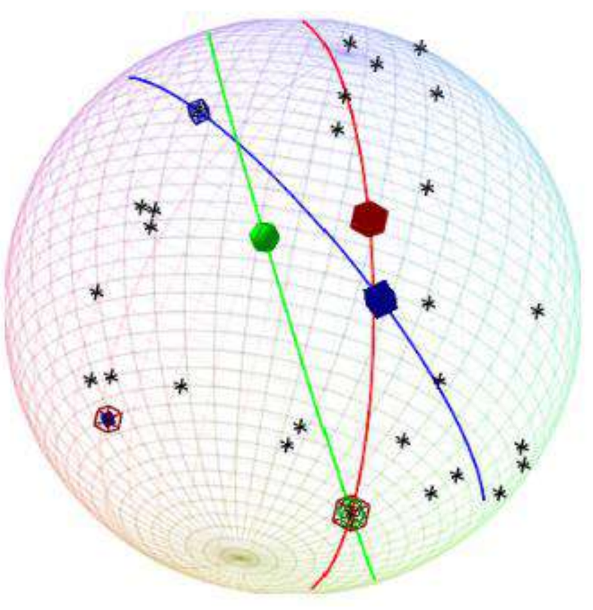
Barycentric subspace analysis (BSA):

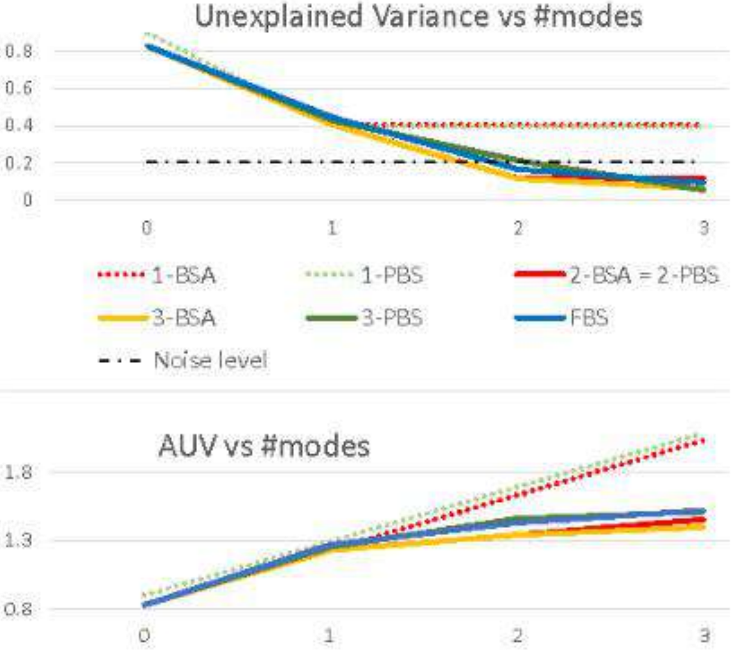
- Energy on flags: Accumulated Unexplained Variance
  → produce the right ordered flags of subspaces in Euclidean spaces

X.P. Barycentric Subspace Analysis on Manifolds [arXiv:1607.02833]

Algorithms: Sample-limited inference

- Sample-limited Fréchet mean / template [Lepore et al 2008]
- SL First geodesic mode [Feragen et al. 2013, Zhai et al 2016]
- Higher orders: challenging with PGA… but not with BSA
  - **FBS: Forward Barycentric Subspace**
  - **k-PBS: Pure Barycentric Subspace with backward ordering**
  - **k-BSA: Barycentric Subspace Analysis up to order k**





Geometric statistics: open problems

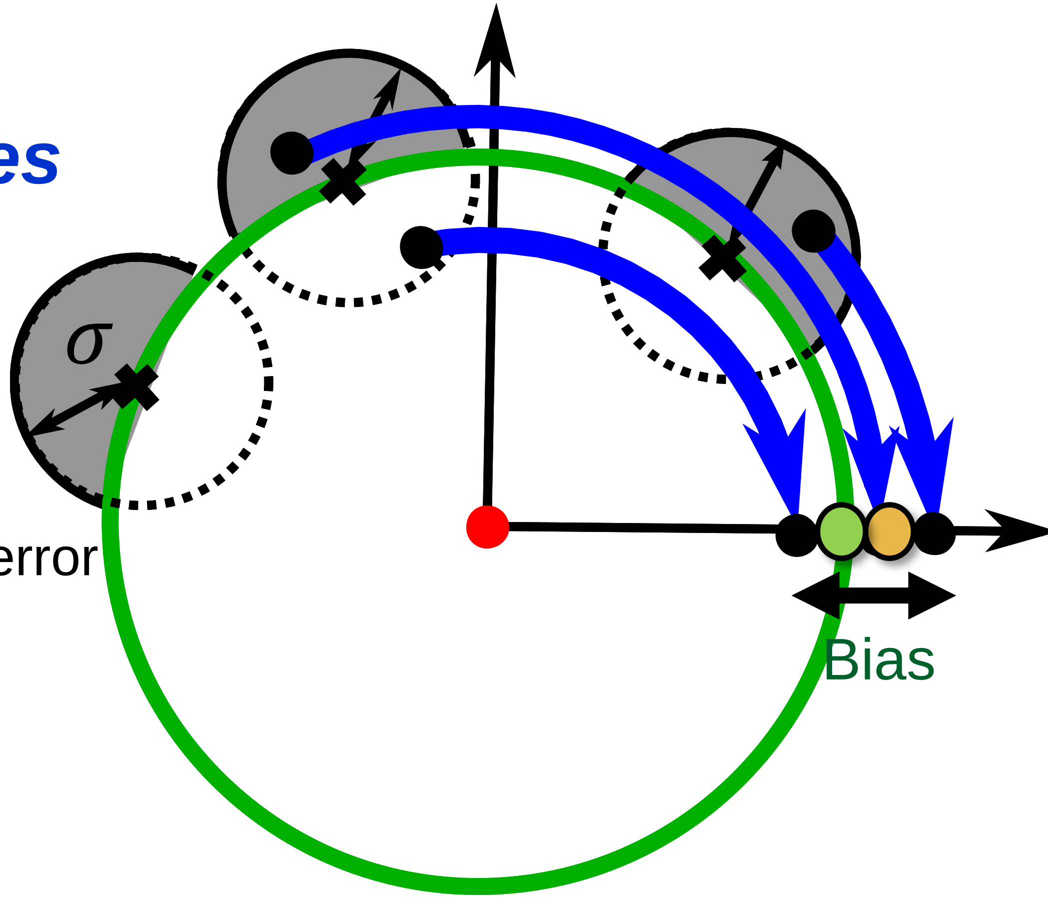
Riemannian statistics

- Advanced statistics (efficiency, Cramer-Rao, CLT [H. Le, Huckemann])
- Which metric? (least-square ~ isotropic noise)
- Completeness issues (geodesic shooting, projection)
- Boundaries, corners (e.g. pure states in qbits?)

Other geometric structures

- Affine connection spaces?
  (bi-invariance on Lie groups with Cartan-Schouten connections)
- Gluing manifolds (e.g. trees)
  Log-map well defined [D. Barden, H. Le, 2017]
- Bundles and quotient spaces:
  stratification if the isotropy group changes: regular strata not sufficient

Noise in top space =
Bias in quotient spaces



σ^2: variance of measurement error

The curvature of the **template shape's orbit and presence of noise** creates bias

Theorem [Miolane et al. (2016)]: Bias of estimator $\hat{T}$ of the template T

$$\mathrm{Bias}(\hat{T}, T) = \frac{\sigma^2}{2} \boldsymbol{H}(T) + \mathcal{O}(\sigma^4)$$

where $\mathbf{H}(T)$ **: mean curvature vector of template's orbit**

References on Template bias with in quotient space

- **Manifold of finite dimension, isometric action, asymptotic for $\sigma \to 0$, correction using bootstrap**
  N. Miolane, S. Holmes, X. Pennec. Template shape computation: correcting an asymptotic bias. SIAM Journal of Imaging Science, 10(2):808-844, 2017.
  Web applications using Shiny (R): http://nmiolane.shinyapps.io/shinyPlane/
  http://nmiolane.shinyapps.io/shinySphere /

- **Hilbert of infinite dimension, proof of bias for $\sigma > 0$, asymptotic for $\sigma \to \infty$, isometric action:**
  L. Devilliers, S. Allasonnière, A. Trouvé and X. Pennec. Template estimation in computational anatomy: Fréchet means in top and quotient spaces are not consistent. SIAM Journal of Imaging Science, 10(3):1139-1169, 2017.

- **Hilbert of infinite dimension, non-isometric action:**
  L. Devilliers, S. Allassonnière, A. Trouvé, and X. Pennec. Inconsistency of Template Estimation by Minimizing of the Variance/Pre-Variance in the Quotient Space. Entropy, 19(6):28, June 2017.

- **Quantification of bias? When and how to correct it?**

References on Barycentric Subspace Analysis

- **Barycentric Subspace Analysis on Manifolds [arXiv:1607.02833]**
  https://hal.archives-ouvertes.fr/hal-01343881
- **Barycentric Subspaces and Affine Spans in Manifolds**
  X. Pennec. Geometric Science of Information GSI'2015, Oct 2015, Palaiseau, France. Proceedings of Geometric Science of Information GSI'2015. Springer LNCS 9389, pp.12-21, 2015. http://dx.doi.org/10.1007/978-3-319-25040-3_2 and https://hal.inria.fr/hal-01164463
  - Warning: change of denomination since this paper: EBS →affine span
- **Barycentric Subspaces Analysis on Spheres**
  X. Pennec. Mathematical Foundations of Computational Anatomy (MFCA'15), Oct 2015, Munich, Germany. pp.71-82, 2015. https://hal.inria.fr/hal-01203815
- **Barycentric subspace analysis: a new symmetric group-wise paradigm for cardiac motion tracking**
  Marc-Michel Rohé, Maxime Sermesant and Xavier Pennec. Proc of MICCAI 2016, Athens, LNCS 9902, p.300-307, Oct 2016. To appear in Medical Image Analysis.

References for Statistics on Manifolds and Lie Groups

Statistics on Riemannnian manifolds

- Xavier Pennec. Intrinsic Statistics on Riemannian Manifolds: Basic Tools for Geometric Measurements. Journal of Mathematical Imaging and Vision, 25(1):127-154, July 2006. http://www.inria.fr/sophia/asclepios/Publications/Xavier.Pennec/Pennec.JMIV06.pdf

Invariant metric on SPD matrices and of Frechet mean to define manifold-valued image processing algorithms

- Xavier Pennec, Pierre Fillard, and Nicholas Ayache. A Riemannian Framework for Tensor Computing. International Journal of Computer Vision, 66(1):41-66, Jan. 2006. http://www.inria.fr/sophia/asclepios/Publications/Xavier.Pennec/Pennec.IJCV05.pdf

Bi-invariant means with Cartan connections on Lie groups

- Xavier Pennec and Vincent Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In Frederic Barbaresco, Amit Mishra, and Frank Nielsen, editors, Matrix Information Geometry, pages 123-166. Springer, May 2012. http://hal.inria.fr/hal-00699361/PDF/Bi-Invar-Means.pdf

Cartan connexion for diffeomorphisms:

- Marco Lorenzi and Xavier Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. International Journal of Computer Vision, 105(2), November 2013 https://hal.inria.fr/hal-00813835/document