Pontryagin calculus in Riemannian geometry

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Geometric Science of Information - GSI 2015
Palaiseau (France), Friday 30 october 2015

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Outlook

Introduction

1) Pontryagin framework for differential equations

2) Pontryagin hamiltonian

3) Riemannian metric for robotics applications

4) Optimal dynamics

5) Intrinsic evolution of the generalized force

Conclusion

Introduction

Dynamics of articulated systems

The choice of the Lagrangian is directly linked
to the conservation of energy.

Euler-Lagrange methodology applied to the
kinetic and potential energies.

System of second order ordinary differential equations
for the motion.

These equations are identical to those deduced
from the fundamental principle of dynamics.

Introduction *(ii)*

Configuration parameters:

their choice does not affect the energy value.

The kinetic energy is a positive definite quadratic form
with respect to the configuration parameters derivatives.
Its coefficients are ideal candidates to define and create a
Riemannian metric structure on the configuration space.

The Euler-Lagrange equations have a contravariant tensorial
nature and highlight the covariant derivatives with respect
to time with the introduction of the Christoffel symbols.

Control of articulated robot

How to choose a Hamiltonian and a cost function ?
Here the presence of the Riemann structure is sound.
It enables a cost function invariant when coordinates change.

References

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Pontryagin framework for differential equations

- Dynamical system : state vector $y(t\,;\,\lambda(\bullet))$
  System controlled by a set of variables $\lambda(t)$
  First order ordinary differential equation:
  $$(1) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t),\, \lambda(t),\, t)\,.$$
  Initial condition: $y(0\,;\,\lambda(\bullet)) = x\,.$
  Optimal solution : minimize the following cost function J :
  $$J(\lambda(\bullet)) \equiv \int_0^T g\big(y(t),\, \lambda(t),\, t\big)\,\mathrm{d}t\,.$$
- Pontryagin's main idea : consider the differential equation (1) as a constraint satisfied by the variable y.
  Lagrange multiplier $p = p(t)$ associated to the constraint (1).
  This new variable is a covariant vector function of time
  Global Lagrangian functional :

$$\mathcal{L}(y,\, \lambda,\, p) \equiv \int_0^T g(y,\, \lambda,\, t)\,\mathrm{d}t + \int_0^T p(t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} - f(y,\, \lambda,\, t)\right)\mathrm{d}t\,.$$

Adjoint equations

- Proposition 1.

  If the Lagrange multiplier $p(t)$ satisfies the adjoint equations,

  $$\frac{\mathrm{d}p}{\mathrm{d}t} + p\,\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \;=\; 0$$

  and the final condition : $p(T) \;=\; 0\,,$

  the variation δJ of the cost function

  for a given variation $\delta\lambda$ of the paramater

  is given by the relation

  $$\delta J \;=\; \int_0^T \Big[\,\frac{\partial g}{\partial \lambda} - p\,\frac{\partial f}{\partial \lambda}\,\Big]\,\delta\lambda(t)\,\mathrm{d}t\,.$$

  At the optimum this variation is identically null :

  Pontryagin optimality condition: $\dfrac{\partial g}{\partial \lambda} - p\,\dfrac{\partial f}{\partial \lambda} \;=\; 0\,.$

Pontryagin hamiltonian

- Hamiltonian : $\mathcal{H}(p, y, \lambda) \equiv p f - g$
  Optimal Hamiltonian $H(p, y) \equiv \mathcal{H}(p, y, \lambda^*)$
  for $\lambda(t) = \lambda^*(t)$ equal to the optimal value
  associated to the optimal condition $J(\lambda^*) \leq J(\lambda) \quad \forall \lambda$

- Proposition 2. Symplectic form of the dynamic equations.
  the "forward" differential equation
  $$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t), \lambda(t), t) .$$
  and the "backward" adjoint differential equation
  $$\frac{\mathrm{d}p}{\mathrm{d}t} + p \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} = 0$$
  take the symplectic form:
  $$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial H}{\partial p} , \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial y} .$$

Riemannian metric

- Dynamical system

  parameterized by a finite number of functions $q^j(t)$
  Set of all states Q: $q \equiv \{q^j\}$

  The kinetic energy K is a positive definite quadratic form
  of the time derivatives $\dot{q}^j$ for each state $q \in Q$.
  The coefficients of this quadratic form define a
  mass tensor $M(q)$.

  The mass tensor is composed by a nonlinear
  regular function of the state $q \in Q$.

  It contains the mechanical caracteristics
  of mass and inertia of the articulated system.

- We have $K(q, \dot{q}) \equiv \dfrac{1}{2} \sum_{k\,\ell} M_{k\ell}(q)\, \dot{q}^k\, \dot{q}^\ell \,.$

Riemannian metric *(ii)*

- The mass tensor $M(q)$
  is symmetric and positive definite for each state q.
  Consider the Riemannian metric g
  defined by the mass tensor M.
  (Lazrak and Vallée (1995), Siebert (2012))
  We set: $g_{k\ell}(q) \equiv M_{k\ell}(q)$.

- Riemannian manifold structure for the space of states Q
  classical geometrical tools of Riemannian geometry:
  Covariant space derivation $\partial_j \equiv \frac{\partial}{\partial q^j}$
  Contravariant space derivation ∂^j: $< \partial^j , \partial_k >= \delta^j_k$
  Component j, ℓ of the inverse mass tensor M^{-1}: $M^{j\ell}$,

$$M_{ij} \; M^{j\ell} = \delta^\ell_i$$

Riemannian metric *(iii)*

- Connection $\Gamma^j_{ik} = \frac{1}{2}\, M^{j\ell} \left(\partial_i M_{\ell k} + \partial_k M_{\ell i} - \partial_\ell M_{ik}\right),$

$$\Gamma^j_{ki} = \Gamma^j_{ik},$$

$$\mathrm{d}\,\partial_j = \Gamma^\ell_{jk}\, \mathrm{d}q^k\, \partial_\ell, \qquad \mathrm{d}\,\partial^j = -\Gamma^j_{k\ell}\, \mathrm{d}q^k\, \partial^\ell,$$

Relations between covariant components φ_j
and contravariant components φ^k of a vector field:

$$\varphi_j = M_{jk}\, \varphi^k, \;\; \varphi^k = M^{kj}\, \varphi_j$$

Covariant derivation of a vector field $\varphi \equiv \varphi^j\, \partial_j$:

$$\mathrm{d}\varphi = \left(\partial_\ell \varphi^j + \Gamma^j_{\ell k}\, \varphi^k\right) \mathrm{d}q^\ell\, \partial_j$$

Covariant derivation of a covector field $\varphi \equiv \varphi_\ell\, \partial^\ell$:

$$\mathrm{d}\varphi = \left(\partial_k \varphi_\ell - \Gamma^j_{k\ell}\, \varphi_j\right) \mathrm{d}q^k\, \partial^\ell$$

Riemannian metric *(iv)*

- Ricci identities

$$\partial_j M_{k\ell} = \Gamma^p_{jk} M_{\ell p} + \Gamma^p_{j\ell} M_{kp}$$

$$\partial_j M^{k\ell} = -\Gamma^k_{jp} M^{p\ell} - \Gamma^\ell_{jp} M^{pk}$$

Gradient of a scalar field: $\mathrm{d}V = \partial_\ell V \, \mathrm{d}q^\ell = < \nabla V \, , \mathrm{d}q^j \, \partial_j >$

$$\nabla V = \partial_\ell V \, \partial^\ell$$

Gradient of a covector field $\varphi = \varphi_\ell \, \partial^\ell$:

$$\mathrm{d}\varphi \equiv < \nabla\varphi \, , \mathrm{d}q^j \, \partial_j > \qquad \nabla\varphi = \left(\partial_k \varphi_\ell - \Gamma^j_{k\ell} \, \varphi_j \right) \partial^k \, \partial^\ell$$

Second order gradient of a scalar field V: $\nabla^2 V = \nabla(\nabla V)$

$$\nabla^2 V = \left(\partial_k \partial_\ell V - \Gamma^j_{k\ell} \, \partial_j V \right) \partial^k \, \partial^\ell$$

- Components $R^j_{ik\ell}$ of the Riemann tensor:

$$R^j_{ik\ell} \equiv \partial_\ell \Gamma^j_{ik} - \partial_k \Gamma^j_{i\ell} + \Gamma^p_{ik} \, \Gamma^j_{p\ell} - \Gamma^p_{i\ell} \, \Gamma^j_{pk}$$

Anti-symmetry of the Riemann tensor: $\quad R^j_{ik\ell} = -R^j_{i\ell k}$.

Riemannian form of the Euler-Lagrange equations

- Proposition 3.
  In the presence of an external potential $V = V(q)$,
  the Lagrangian $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$ allows to write
  the equations of motion in the classical Euler-Lagrange form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) = \frac{\partial L}{\partial q^i}$$

  These equations take also the Riemannian form:

$$M_{k\ell}\,\big(\ddot{q}^\ell + \Gamma^\ell_{ij}\,\dot{q}^i\,\dot{q}^j\big) + \partial_k V = 0 \quad .$$

- Equations of motion when a mechanical forcing control u
  is present (forces and torques typically):

$$\ddot{q}^j + \Gamma^j_{k\ell}\,\dot{q}^k\,\dot{q}^\ell + M^{j\ell}\,\partial_\ell V = u^j \quad .$$

  with the contravariant components of the control u
  in the right hand side.

Optimal dynamics

- Cost function

  The space of states Q has a natural Riemannian structure.

  Intrinsic and invariant cost function

  non sensible to the change of coordinates.

  $$(2) \quad J(u) = \frac{1}{2} \int_0^T M_{k\ell}(q)\, u^k\, u^\ell \, \mathrm{d}t \, .$$

  (Rojas Quintero's thesis, 2013)

- The controlled system $\ddot{q}^j + \Gamma^j_{k\ell}\, \dot{q}^k\, \dot{q}^\ell + M^{j\ell}\, \partial_\ell V \; = \; u^j$

  with the cost function (2) is of type

  $$\frac{\mathrm{d}Y}{\mathrm{d}t} = f(Y(t), \lambda(t), t)\, , \;\; J(\lambda(\bullet)) \equiv \int_0^T g(Y(t), \lambda(t), t)\, \mathrm{d}t$$

  with

  $$Y = \{q^j \, , \, \dot{q}^j\}\, , \qquad f = \{Y_2^j\, , -\Gamma^j_{k\ell}\, \dot{q}^k\, \dot{q}^\ell - M^{j\ell}\, \partial_\ell V + u^j\}\, ,$$

  $$\lambda = \{u^k\}\, , \qquad g = \frac{1}{2} M_{k\ell}(Y_1)\, u^k\, u^\ell \, .$$

Optimal dynamics *(ii)*

- Lagrange multipliers of the Pontryagin method :
  $P \,=\, \{p_j \,,\, \xi_j\}$
- Hamiltonian $\mathcal{H}(Y,\, P,\, \lambda)$
  function of state Y, adjoint P
  and control variable $\lambda \,=\, \{u^k\}$
  We have: $\mathcal{H}(Y,\, P,\, \lambda) \,=$
  $$p_j \, \dot{q}^j + \xi_j \left[-\Gamma^j_{k\ell} \, \dot{q}^k \, \dot{q}^\ell - M^{j\ell} \, \partial_\ell V + u^j \right] - \frac{1}{2} \, M_{k\ell}(Y_1) \, u^k \, u^\ell \,.$$
- Proposition 4. Interpretation of one adjoint state.
  When the cost function $J(u) \,\equiv\, \frac{1}{2} \int_0^T M_{k\ell}(q) \, u^k \, u^\ell \, \mathrm{d}t$
  is stationary, the adjoint state ξ^j is exactly equal to the
  applied force (and torque!) u^j : $\xi^j \,=\, u^j$.

Intrinsic evolution of the generalized force

- Covector ξ : $\xi = \xi_j \, \partial^j$.
  We have the following result:

- Proposition 5.
  Covariant evolution equation of the optimal force.
  With the above notations and hypotheses,
  the forces and torques u satisfy the following time evolution:
  $$\Big(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} \Big)_j + R^i_{k\ell j} \, \dot{q}^k \, \dot{q}^\ell \, u_i + \big(\nabla^2_{jk} V \big) \, u^k \, = \, 0 \ .$$

(Rojas Quintero's thesis, and Vallée *et al.*, 2013).

Conclusion

The control optimization of a robotic system shows how important is the introduction of an appropriate geometric structure.

Riemannian geometry favors the metric associated to the kinetic energy.

The systems have a well-defined Riemannian tensorial nature:
contravariant for the equation of motion
covariant for the equation of the control variables.

Mechanical interpretation of the Pontryagin's adjoint states.

Second order covariant derivatives and Riemann curvature tensor for the equation of forces and torques.

Numerically stable method when discretization is considered.
Future numerical developments: juggle between two coupled systems of second-order ordinary differential equations.

Thank you for your attention !