

Information Algebras and their Applications

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Based on:

- M. Marcolli, R. Thorngren, *Thermodynamic semirings*, J. Noncommut. Geom. 8 (2014), no. 2, 337–392
- M. Marcolli, N. Tedeschi, *Entropy algebras and Birkhoff factorization*, J. Geom. Phys. 97 (2015) 243–265

Min-Plus Algebra (Tropical Semiring)

min-plus (or tropical) semiring $\mathbb{T} = \mathbb{R} \cup \{\infty\}$

- operations \oplus and \odot

$$x \oplus y = \min\{x, y\} \quad \text{with identity } \infty$$

$$x \odot y = x + y \quad \text{with identity } 0$$

- operations \oplus and \odot satisfy:
 - associativity
 - commutativity
 - left/right identity
 - distributivity of product \odot over sum \oplus

Thermodynamic semirings $\mathbb{T}_{\beta,S} = (\mathbb{R} \cup \{\infty\}, \oplus_{\beta,S}, \odot)$

- deformation of the tropical addition $\oplus_{\beta,S}$

$$x \oplus_{\beta,S} y = \min_p \left\{ px + (1-p)y - \frac{1}{\beta} S(p) \right\}$$

β thermodynamic inverse temperature parameter

$S(p) = S(p, 1-p)$ binary information measure, $p \in [0, 1]$

- for $\beta \rightarrow \infty$ (zero temperature) recovers unperturbed idempotent addition \oplus
- multiplication $\odot = +$ is undeformed
- for $S =$ Shannon entropy considered first in relation to \mathbb{F}_1 -geometry in
 - A. Connes, C. Consani, *From monoids to hyperstructures: in search of an absolute arithmetic*, arXiv:1006.4810

Khinchin axioms $\text{Sh}(p) = -C(p \log p + (1 - p) \log(1 - p))$

• Axiomatic characterization of Shannon entropy $S(p) = \text{Sh}(p)$

① symmetry $S(p) = S(1 - p)$

② minima $S(0) = S(1) = 0$

③ extensivity

$$S(pq) + (1 - pq)S(p(1 - q)/(1 - pq)) = S(p) + pS(q)$$

• correspond to **algebraic properties** of semiring $\mathbb{T}_{\beta, S}$

① commutativity of $\oplus_{\beta, S}$

② left and right identity for $\oplus_{\beta, S}$

③ associativity of $\oplus_{\beta, S}$

$\Rightarrow \mathbb{T}_{\beta, S}$ commutative, unital, associative **iff** $S(p) = \text{Sh}(p)$

Khinchin axioms n -ary form

Given S as above, define $S_n : \Delta_{n-1} \rightarrow \mathbb{R}_{\geq 0}$ by

$$S_n(p_1, \dots, p_n) = \sum_{1 \leq j \leq n-1} \left(1 - \sum_{1 \leq i < j} p_i\right) S\left(\frac{p_j}{1 - \sum_{1 \leq i < j} p_i}\right).$$

Then Khinchin axioms:

- 1 (Continuity) $S(p_1, \dots, p_n)$ continuous in $(p_1, \dots, p_n) \in \Delta_n$ simplex
- 2 (Maximality) $S(p_1, \dots, p_n)$ maximum at the uniform $p_i = 1/n$
- 3 (Additivity/Extensivity) $p_i = \sum_{j=1}^{m_i} p_{ij}$ then

$$S(p_{11}, \dots, p_{nm_n}) = S(p_1, \dots, p_n) + \sum_{i=1}^n p_i S\left(\frac{p_{i1}}{p_i}, \dots, \frac{p_{im_i}}{p_i}\right);$$

- 4 (Expandability) Δ_n face in Δ_{n+1}

$$S(p_1, \dots, p_n, 0) = S(p_1, \dots, p_n)$$

Shannon entropy case: equivalent form of $\oplus_{\beta, S}$

$$x \oplus_{\beta, S} y = -\beta^{-1} \log \left(e^{-\beta x} + e^{-\beta y} \right)$$

leads to relation with Maslov dequantization and tropical geometry

Rényi entropy:

$$\text{Ry}_\alpha(p_1, \dots, p_n) := \frac{1}{1-\alpha} \log \left(\sum_i p_i^\alpha \right)$$

$$\lim_{\alpha \rightarrow 1} \text{Ry}_\alpha(p_1, \dots, p_n) = \text{Sh}(p_1, \dots, p_n)$$

- lack of associativity of $x \oplus_S y$, when $S = \text{Ry}_\alpha$

$$\text{Ry}_\alpha(p) = \frac{1}{1-\alpha} \log(p^\alpha + (1-p)^\alpha)$$

measured by the transformation $(p_1, p_2, p_3) \mapsto (p_3, p_2, p_1)$

Non-extensive thermodynamics:

- gas of particles with chemical potentials $\log x$ and $\log y$ and Hamiltonian (p mole fraction)

$$\mathcal{H} = p \log x + (1 - p) \log y$$

- partition function $Z = e^{-F_{\text{eq}}}$ with F_{eq} equilibrium value of free energy at temperature $T = 1/\beta$

$$x \oplus_{\beta, S} y = \max_p (e^{TS(p) + p \log x + (1-p) \log y})$$

partition sum of a two state system with energies x and y

- Extensive thermodynamics: independent subsystems A and B , combined system $A \star B$

$$S(A \star B) = S(A) + S(B)$$

- Non-extensive deformations (Tsallis)

$$S_q(A \star B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B)$$

Tsallis entropy:

$$T_{S_\alpha}(p) = \frac{1}{\alpha - 1} (1 - p^\alpha - (1 - p)^\alpha)$$

reproduces Shannon entropy $\alpha \rightarrow 1$

- Tsallis entropy uniquely determined by symmetry

$S(p) = S(1 - p)$, minima $S(0) = S(1) = 0$, and α -deformed extensivity

$$S(p_1) + (1 - p_1)^\alpha S\left(\frac{p_2}{1 - p_1}\right) = S(p_1 + p_2) + (p_1 + p_2)^\alpha S\left(\frac{p_1}{p_1 + p_2}\right)$$

Tsallis thermodynamic semiring

- Tsallis thermodynamic semiring: commutativity, unitarity and associativity of α -deformed $\oplus_{\beta, S, \alpha}$

$$x \oplus_{\beta, S, \alpha} y = \min_p \left\{ p^\alpha x + (1 - p)^\alpha y - \frac{1}{\beta} \text{Ts}_\alpha(p) \right\}$$

- **General idea:** transform axiomatic characterizations of various entropy functionals into algebraic properties of corresponding thermodynamic deformations of min-plus algebras

Entropy Operads

- Operad: objects $\mathcal{C}(j)$ in a symmetric monoidal category: parameter space of j -ary operations with composition maps

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \rightarrow \mathcal{C}(j_1 + \cdots + j_k)$$

associative, unital, and equivariant under permutations

- \mathcal{C} -algebra A : an object with Sym_j -equivariant maps

$$\mathcal{C}(j) \otimes A^j \rightarrow A,$$

thought of as actions, associative and unital

- operad \mathcal{P} of probabilities on finite sets $\mathcal{P}(j) = \Delta_j$ simplex

$$\gamma((p_i)_{i \in j} \otimes (q_{1l})_{l \in k_0} \otimes \cdots \otimes (q_{jl})_{l \in k_{j-1}}) = (p_i q_{il})_{l \in k_i, i \in j} \in \mathcal{C}(k_0 + \cdots + k_{j-1})$$

composite of subsystems

Information Algebras (over the entropy operad)

• object $\mathbb{R}_{\geq 0}$, morphisms $x \in \mathbb{R}_{\geq 0}$; action of operad \mathcal{P} : maps S from finite probabilities to non-negative real number with

① For $p \in \mathcal{P}(n)$ and $q_i \in \mathcal{P}(m_i)$

$$S(p \circ (q_1, \dots, q_n)) = S(p) + \sum_i p_i S(q_i);$$

② $S((1)) = 0$;

③ for $p \in \mathcal{P}(n)$ and $\sigma \in \text{Sym}_n$

$$S(\sigma p) = S(p)$$

④ $S : \mathcal{P}(n) \rightarrow \mathbb{R}_{\geq 0}$ continuous

Characterizes entropy functionals

• related work of Tobias Fritz, and J. Baez, T. Fritz, T. Leinster

von Neumann entropy and the tropical trace

- convex set of density matrices

$$\mathcal{M}^{(N)} = \{\rho \in M_{N \times N}(\mathbb{C}) \mid \rho^* = \rho, \rho \geq 0, \text{Tr}(\rho) = 1\}$$

- von Neumann entropy

$$\mathcal{N}(\rho) = -\text{Tr}(\rho \log \rho), \quad \text{for } \rho \in \mathcal{M}^{(N)}$$

Shannon entropy in diagonal case

- matrices $M_{N \times N}(\mathbb{T})$ over $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$$(A \oplus B)_{ij} = \min\{A_{ij}, B_{ij}\} \quad \text{and} \quad (A \odot B)_{ij} = \oplus_k A_{ik} \odot B_{kj} = \min_k \{A_{ik} + B_{kj}\}$$

- tropical trace $\text{Tr}^\oplus(A) = \min_i \{A_{ii}\}$

Entropical trace: thermodynamic deformation of tropical trace

$$\text{Tr}_{\beta, S}^\oplus(A) := \min_{\rho \in \mathcal{M}^{(N)}} \{\text{Tr}(\rho A) - \beta^{-1} S(\rho)\}$$

Tr in the right-hand-side is the *ordinary* trace



Kullback–Leibler divergence and von Neumann entropical trace

- relative entropy (Kullback–Leibler divergence)

$$S(\rho||\sigma) = \text{Tr}(\rho(\log \rho - \log \sigma))$$

- von Neumann deformation and relative entropy
(for $A = A^*$, $A \geq 0$)

$$\text{Tr}(\rho A) - \beta^{-1} \mathcal{N}(\rho) = \frac{1}{\beta} S(\rho||\sigma_{\beta,A}) - \frac{1}{\beta} \log Z_A(\beta)$$

$$\sigma_{\beta,A} = \frac{e^{-\beta A}}{Z_A(\beta)} \quad \text{with} \quad Z_A(\beta) = \text{Tr}(e^{-\beta A})$$

- von Neumann entropical trace (for $A = A^*$, $A \geq 0$)

$$\text{Tr}_{\beta,\mathcal{N}}^{\oplus}(A) = -\frac{\log Z_A(\beta)}{\beta}$$

with $Z_A(\beta) = \text{Tr}(e^{-\beta A})$: rhs above is Helmholtz free energy

Algebraic Renormalization (Connes–Kreimer)

- Feynman graphs of a quantum field theory form a commutative Hopf algebra \mathcal{H}
- Feynman rules: morphism ϕ of *commutative algebras* from \mathcal{H} to a target algebra \mathcal{R} (e.g. Laurent series)
- subtraction of infinities: Birkhoff factorization procedure
- multiplicative decomposition of ϕ into divergences (counterterms) and renormalized valued, obtained using
 - coproduct Δ of Hopf algebra \mathcal{H}
 - Rota-Baxter operator (e.g. pole subtraction) on \mathcal{R}
- in physics BPHZ renormalization

Connes–Kreimer Hopf algebra of Feynman graphs \mathcal{H}

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom): Ward identities as Hopf ideals

Rota–Baxter algebras (Ebrahimi-Fard, Guo, Kreimer)

- **Rota–Baxter algebra** of weight $\lambda = -1$: \mathcal{R} commutative unital algebra; $\mathcal{T} : \mathcal{R} \rightarrow \mathcal{R}$ linear operator with

$$\mathcal{T}(x)\mathcal{T}(y) = \mathcal{T}(x\mathcal{T}(y)) + \mathcal{T}(\mathcal{T}(x)y) + \lambda\mathcal{T}(xy)$$

- Example: \mathcal{T} = projection onto polar part of Laurent series
- \mathcal{T} determines splitting $\mathcal{R}_+ = (1 - \mathcal{T})\mathcal{R}$, $\mathcal{R}_- =$ unitization of $\mathcal{T}\mathcal{R}$; both \mathcal{R}_\pm are algebras
- **Feynman rule** $\phi : \mathcal{H} \rightarrow \mathcal{R}$ commutative algebra homomorphism from CK Hopf algebra \mathcal{H} to Rota–Baxter algebra \mathcal{R} weight -1

$$\phi \in \text{Hom}_{\text{Alg}}(\mathcal{H}, \mathcal{R})$$

- **Note:** ϕ does *not* know that \mathcal{H} Hopf and \mathcal{R} Rota–Baxter, only commutative algebras

- **Birkhoff factorization** $\exists \phi_{\pm} \in \text{Hom}_{\text{Alg}}(\mathcal{H}, \mathcal{R}_{\pm})$

$$\phi = (\phi_- \circ S) \star \phi_+$$

where $\phi_1 \star \phi_2(x) = \langle \phi_1 \otimes \phi_2, \Delta(x) \rangle$

- Connes-Kreimer inductive formula for Birkhoff factorization:

$$\phi_-(x) = -\mathcal{T}(\phi(x) + \sum \phi_-(x')\phi(x''))$$

$$\phi_+(x) = (1 - \mathcal{T})(\phi(x) + \sum \phi_-(x')\phi(x''))$$

where $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$

- Bogolyubov-Parshchuk preparation

$$\tilde{\phi}(x) = \phi(x) + \sum \phi_-(x')\phi(x'')$$

Renormalization and Computation (Manin)

proposal for a “renormalization of the halting problem”

- Idea: treat noncomputable functions like infinities in QFT
- Renormalization = extraction of finite part from divergent Feynman integrals; extraction of “computable part” from noncomputables
- First step: build a Hopf algebra (flow charts, partial recursive functions) and a Feynman rule that detects the presence of noncomputability (infinities)
- Second step: BPHZ type subtraction procedure **with values in a min-plus or max-plus algebra** (computing time, memory size)
- Third step: meaning of the “renormalized part” and of the “divergences part” of the Birkhoff factorization in terms of theory of computation

Semirings of functions

- min-plus semirings $\mathbb{S} = C(X, \mathbb{T})$ with pointwise \oplus, \odot
- thermodynamic deformations $\mathbb{S}_{\beta, S} = C(X, \mathbb{T}_{\beta, S})$ with pointwise $\oplus_{\beta, S}, \odot$

Logarithmically related pairs $(\mathcal{R}, \mathbb{S})$

- \mathcal{R} commutative ring (algebra); \mathbb{S} min-plus semiring; with formal logarithm bijective map $\mathcal{L} : \text{Dom}(\mathcal{L}) \subset \mathcal{R} \rightarrow \mathbb{S}$

$$\mathcal{L}(ab) = \mathcal{L}(a) + \mathcal{L}(b) = \mathcal{L}(a) \odot \mathcal{L}(b)$$

- thermodynamic deformation (Shannon entropy)

$$f_1 \oplus_{\beta, S} f_2 = -\beta^{-1} \log(E(-\beta f_1) + E(-\beta f_2))$$

with $E : \mathbb{S} \rightarrow \text{Dom}(\mathcal{L}) \subset \mathcal{R}$ inverse of \mathcal{L}

Examples

- $\mathcal{R} = C(X, \mathbb{R})$ and $\text{Dom}(\mathcal{L}) \subset \mathcal{R}$ given by $C(X, \mathbb{R}_+^*)$ with $\mathcal{L}(a) = -\beta^{-1} \log(a)$

$$f_1 \oplus_{\beta, S} f_2 = -\beta^{-1} \log(e^{-\beta f_1} + e^{-\beta f_2})$$

is $\mathbb{S}_{\beta, S} = C(X, \mathbb{T}_{\beta, S})$ with $S = \text{Sh}$

- $\mathcal{R} = \mathbb{Q}[[t]]$ ring of formal power series, $\text{Dom}(\mathcal{L}) \subset \mathcal{R}$ power series $\alpha(t) = \sum_{k \geq 0} a_k t^k$ with $a_0 = 1$, with formal log

$$\mathcal{L}(1 + \alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \alpha^k$$

$$\alpha_1 \oplus_{\beta, S} \alpha_2 = \beta^{-1} \mathcal{L}(E(-\beta \alpha_1) + E(-\beta \alpha_2))$$

formal exponential $E(\gamma) = \sum_{k \geq 0} \gamma^k / k!$

min-plus valued characters (algebraic Feynman rules)

- \mathcal{H} commutative Hopf algebra; \mathbb{S} be a min-plus semiring
- $\psi : \mathcal{H} \rightarrow \mathbb{S}$ satisfying $\psi(1) = 0$ and

$$\psi(xy) = \psi(x) + \psi(y), \quad \forall x, y \in \mathcal{H}$$

- main idea: “arithmetic of orders of magnitude” $\epsilon \rightarrow 0$
 - leading term in $\epsilon^\alpha + \epsilon^\beta$ is $\epsilon^{\min\{\alpha, \beta\}}$
 - leading term of $\epsilon^\alpha \epsilon^\beta$ is $\epsilon^{\alpha + \beta}$
- model characters and Birkhoff factorization on “order of magnitude” version of usual ones

convolution of min-plus characters

$$(\psi_1 \star \psi_2)(x) = \min\{\psi_1(x^{(1)}) + \psi_2(x^{(2)})\} = \bigoplus (\psi_1(x^{(1)}) \odot \psi_2(x^{(2)}))$$

minimum over all pairs $(x^{(1)}, x^{(2)})$ in coproduct
 $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$ in Hopf algebra \mathcal{H}

Birkhoff factorization of a min-plus character ψ

$$\psi_+ = \psi_- \star \psi$$

\star convolution product, ψ_{\pm} satisfying $\psi_{\pm}(xy) = \psi_{\pm}(x) + \psi_{\pm}(y)$

Note: does not require antipode, works also for \mathcal{H} bialgebra

Rota–Baxter semirings

- \mathbb{S} be a min-plus semiring, map $T : \mathbb{S} \rightarrow \mathbb{S}$ is \oplus -additive if monotone, $T(a) \leq T(b)$ for $a \leq b$ (pointwise)

- Rota–Baxter semiring $(\mathbb{S}, \oplus, \odot)$ weight $\lambda > 0$:
exists \oplus -additive map $T : \mathbb{S} \rightarrow \mathbb{S}$ with

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2)) \oplus T(f_1 \odot f_2) \odot \log \lambda$$

- Rota–Baxter semiring $(\mathbb{S}, \oplus, \odot)$ weight $\lambda < 0$:
exists \oplus -additive map $T : \mathbb{S} \rightarrow \mathbb{S}$ with

$$T(f_1) \odot T(f_2) \oplus T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2))$$

Birkhoff factorization in min-plus semirings (weight +1)

- Bogolyubov-Parashchuk preparation

$$\tilde{\psi}(x) = \min\{\psi(x), \psi_-(x') + \psi(x'')\} = \psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')$$

(x', x'') ranges over non-primitive part of coproduct

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$$

- ψ_- defined inductively on lower degree x' in Hopf algebra

$$\begin{aligned}\psi_-(x) &:= T(\tilde{\psi}(x)) = T(\min\{\psi(x), \psi_-(x') + \psi(x'')\}) \\ &= T\left(\psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')\right)\end{aligned}$$

- by \oplus -linearity of T same as

$$\begin{aligned}\psi_-(x) &= \min\{T(\psi(x)), T(\psi_-(x') + \psi(x''))\} \\ &= T(\psi(x)) \oplus \bigoplus T(\psi_-(x') \odot \psi(x''))\end{aligned}$$

- then ψ_+ by convolution

$$\begin{aligned}\psi_+(x) &:= (\psi_- \star \psi)(x) = \min\{\psi_-(x), \psi(x), \psi_-(x') + \psi(x'')\} \\ &= \min\{\psi_-(x), \tilde{\psi}(x)\} = \psi_-(x) \oplus \tilde{\psi}(x)\end{aligned}$$

- **key step:** associativity and commutativity of \oplus and \oplus -additivity of T , plus Rota-Baxter identity weight $+1$ gives

$$\psi_-(xy) = \psi_-(x) + \psi_-(y)$$

hence ψ_+ also as convolution

Thermodynamic Rota–Baxter structures

- $\mathbb{S}_{\beta,S}$ thermodynamic Rota–Baxter semiring weight $\lambda > 0$: there is $\oplus_{\beta,S}$ -additive map $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2)) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log \lambda$$

- $\mathbb{S}_{\beta,S}$ thermodynamic Rota–Baxter semiring weight $\lambda < 0$: there is $\oplus_{\beta,S}$ -additive map $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$

$$T(f_1) \odot T(f_2) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2))$$

like previous case but with \oplus replaced with deformed $\oplus_{\beta,S}$

- $(\mathcal{R}, \mathbb{S})$ logarithmically related pair: $T : \mathbb{S} \rightarrow \mathbb{S}$ determines $\mathcal{T} : \mathcal{R} \rightarrow \mathcal{R}$ with $\mathcal{T}(e^{-\beta f}) := e^{-\beta T(f)}$, for $a = e^{-\beta f}$ in $\text{Dom}(\log) \subset \mathcal{R}$

\mathcal{T} Rota-Baxter weight λ_β on $\mathcal{R} \Leftrightarrow T$ Rota-Baxter weight λ on $\mathbb{S}_{\beta, S}$ with $S = \text{Sh}$ and $\lambda_\beta = \lambda^{-\beta}$, for $\lambda > 0$, or $\lambda_\beta = -|\lambda|^{-\beta}$ for $\lambda < 0$

$$\begin{aligned} \mathcal{T}(e^{-\beta f_1})\mathcal{T}(e^{-\beta f_2}) &= \mathcal{T}(\mathcal{T}(e^{-\beta f_1})e^{-\beta f_2}) + \mathcal{T}(e^{-\beta f_1}\mathcal{T}(e^{-\beta f_2})) \\ &\quad + \lambda_\beta \mathcal{T}(e^{-\beta f_1}e^{-\beta f_2}) \end{aligned}$$

- \mathcal{T} is \mathbb{R} -linear iff T is $\oplus_{\beta, S}$ -linear

Birkhoff factorization in thermodynamic Rota–Baxter semirings (weight +1)

- $T : \mathbb{S}_{\beta, \mathcal{S}} \rightarrow \mathbb{S}_{\beta, \mathcal{S}}$ Rota–Baxter of weight $\lambda = +1$
- Bogolyubov–Parashchuk preparation of $\psi : \mathcal{H} \rightarrow \mathbb{S}_{\beta, \mathcal{S}}$

$$\tilde{\psi}_{\beta, \mathcal{S}}(x) = \psi(x) \oplus_{\beta, \mathcal{S}} \bigoplus_{\beta, \mathcal{S}} \psi_-(x') + \psi(x'')$$

$$= -\beta^{-1} \log \left(e^{-\beta\psi(x)} + \sum e^{-\beta(\psi_-(x') + \psi(x''))} \right)$$

- $\phi_\beta(x) := e^{-\beta\psi(x)}$ in \mathcal{R} : Bogolyubov–Parashchuk preparation
 $\tilde{\phi}_\beta(x) = e^{-\beta\tilde{\psi}(x)}$

$$\tilde{\phi}_\beta(x) := \phi_\beta(x) + \sum \mathcal{T}(\tilde{\phi}_\beta(x'))\phi_\beta(x'')$$

with $\mathcal{T}(e^{-\beta f}) := e^{-\beta \mathcal{T}(f)}$ and $\mathcal{T}(-e^{-\beta f}) := -\mathcal{T}(e^{-\beta f})$

- Birkhoff factorization $\psi_{\beta,+} = \psi_{\beta,-} \star_{\beta} \psi$

$$\psi_{\beta,-}(x) = T(\tilde{\psi}_{\beta}(x)) = -\beta^{-1} \log \left(e^{-\beta T(\psi(x))} + \sum e^{-\beta T(\psi_{-}(x') + \psi(x''))} \right)$$

$$\psi_{\beta,+}(x) = -\beta^{-1} \log \left(e^{-\beta \psi_{\beta,-}(x)} + e^{-\beta \tilde{\psi}_{\beta}(x)} \right)$$

satisfying $\psi_{\beta,\pm}(xy) = \psi_{\beta,\pm}(x) + \psi_{\beta,\pm}(y)$

- in limit $\beta \rightarrow \infty$ thermodynamic Birkhoff factorization converges to min-plus Birkhoff factorization

Example: Witt rings

- commutative ring R , Witt ring $W(R) = 1 + tR[[t]]$: addition is product of formal power series, multiplication determined by

$$(1 - at)^{-1} \star (1 - bt)^{-1} = (1 - abt)^{-1} \quad a, b \in R$$

- injective ring homomorphism $g : W(R) \rightarrow R^{\mathbb{N}}$, ghost coordinates coefficients of

$$t \frac{1}{\alpha} \frac{d\alpha}{dt} = \sum_{r \geq 1} \alpha_r t^r$$

for $\alpha = \exp(\sum_{r \geq 1} \alpha_r t^r / r)$

- component-wise addition and multiplication on $R^{\mathbb{N}}$

- linear operator $\mathcal{T} : R[[t]] \rightarrow R[[t]]$ is Rota–Baxter weight λ iff $\mathcal{T}_W : W(R) \rightarrow W(R)$ defined by $g(\mathcal{T}_W(\alpha)) = \mathcal{T}(g(\alpha))$ satisfies

$$\begin{aligned} \mathcal{T}_W(\alpha_1) \circledast \mathcal{T}_W(\alpha_2) &= \mathcal{T}_W(\alpha_1 \circledast \mathcal{T}_W(\alpha_2)) +_W \mathcal{T}_W(\mathcal{T}_W(\alpha_1) \circledast \alpha_2) \\ &\quad +_W \lambda \mathcal{T}_W(\alpha_1 \circledast \alpha_2) \end{aligned}$$

with $+_W$ addition of $W(R)$ and convolution product

$$\alpha \circledast \gamma := \exp \left(\sum_{n \geq 1} \left(\sum_{r+\ell=n} \alpha_r \gamma_\ell \right) \frac{t^n}{n} \right)$$

for $\alpha = \exp(\sum_{r \geq 1} \alpha_r t^r / r)$ and $\gamma = \exp(\sum_{r \geq 1} \gamma_r t^r / r)$

- Example: $\mathcal{R} = R^{\mathbb{N}}$ Rota–Baxter weight +1

$$\mathcal{T} : (a_1, a_2, \dots, a_n, \dots) \mapsto (0, a_1, a_1 + a_2, \dots, \sum_{k=1}^{n-1} a_k, \dots)$$

resulting Rota–Baxter \mathcal{T}_W weight +1 on Witt ring $W(R)$

$$\mathcal{T}_W(\alpha) = \alpha \circledast \mathbb{I}$$

convolution product with multiplicative unit $\mathbb{I} = (1 - t)^{-1}$

- Hasse–Weil zeta functions of varieties over \mathbb{F}_q

$$Z(X, t) = \exp \left(\sum_{r \geq 1} \#X(\mathbb{F}_{q^r}) \frac{t^r}{r} \right)$$

elements in Witt ring:

$$Z(X \sqcup Y, t) = Z(X, t)Z(Y, t) \quad \text{and} \quad Z(X \times Y, t) = Z(X, t) \star Z(Y, t)$$

Rota–Baxter operator weight +1

$$\mathcal{T}_W(Z(X, t)) = Z(X, t) \circledast Z(\text{Spec}(\mathbb{F}_q), t)$$

Computation examples

- inclusion–exclusion “cost functions”: $\Gamma = \Gamma_1 \cup \Gamma_2$, $\gamma = \Gamma_1 \cap \Gamma_2$

$$\psi(\Gamma) = \psi(\Gamma_1) + \psi(\Gamma_2) - \psi(\gamma)$$

determine $\psi : \mathcal{H} \rightarrow \mathbb{T}$ character $\psi(xy) = \psi(x) + \psi(y)$

- class of machines $\psi_n(\Gamma)$ step-counting function of n -th machine: when it outputs on computation Γ (Hopf algebra of flow charts)
- Rota–Baxter operator weight +1 of partial sums: Bogolyubov–Parashchuk preparation

$$\tilde{\psi}_n(\Gamma) = \min\{\psi_n(\Gamma), \psi_n(\Gamma/\gamma) + \sum_{k=1}^{n-1} \tilde{\psi}_k(\gamma)\}$$

- a graph Γ with $\psi_n(\Gamma) = \infty$ (n -th machine does not halt) can have $\tilde{\psi}_n(\Gamma) < \infty$ if both
 - source of infinity was localized in $\gamma \setminus \partial\gamma$, so $\psi_n(\Gamma/\gamma) < \infty$
 - $\psi_k(\gamma) < \infty$ for all previous machines

“renormalization of computational infinities” in Manin’s sense

Polynomial countability

- in perturbative quantum field theory: graph hypersurfaces

$$X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{A}^{\#E_\Gamma}$$

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin E(T)} t_e$$

sum over spanning trees

- X variety over \mathbb{Z} , reductions X_p over \mathbb{F}_p

$$\text{counting function } N(X, q) := \#X_p(\mathbb{F}_q)$$

Polynomially countable X if counting function polynomial $P_X(q)$

- Question: when are graph hypersurfaces X_Γ polynomially countable? or equivalently complements $Y_\Gamma = \mathbb{A}^{\#E_\Gamma} \setminus X_\Gamma$
- max-plus character $\psi : \mathcal{H} \rightarrow \mathbb{T}_{max}$ with $N(Y_\Gamma, q) \sim q^{\psi(\Gamma)}$ leading order if Y_Γ polynomially countable or $\psi(\Gamma) := -\infty$ if not
- when Y_Γ not polynomially countable

$$\begin{aligned} \tilde{\psi}(\Gamma) &= \max\{\psi(\Gamma), \tilde{\psi}(\gamma) + \psi(\Gamma/\gamma)\} \\ &= \max\{\psi(\Gamma), \sum_{j=1}^N \psi(\gamma_j) + \psi(\gamma_{j-1}/\gamma_j)\} \end{aligned}$$

identifies chains of subgraphs and quotient graphs whose hypersurfaces are polynomially countable

Work in progress: tropical geometry

- tropical polynomial $p : \mathbb{R}^n \rightarrow \mathbb{R}$ piecewise linear

$$p(x_1, \dots, x_n) = \oplus_{j=1}^m a_j \odot x_1^{k_{j1}} \odot \dots \odot x_n^{k_{jn}} =$$

$$\min\{a_1 + k_{11}x_1 + \dots + k_{1n}x_n, a_2 + k_{21}x_1 + \dots + k_{2n}x_n, \dots, a_m + k_{m1}x_1 + \dots + k_{mn}x_n\}.$$

tropical hypersurface where tropical polynomial non-differentiable

- Entropical geometry: thermodynamic deformations of \mathbb{T}

$$p_{\beta, S}(x_1, \dots, x_n) = \oplus_{\beta, S, j} a_j \odot x_1^{k_{j1}} \odot \dots \odot x_n^{k_{jn}} =$$

$$\min_{p=(p_j)} \left\{ \sum_j p_j (a_j + k_{j1}x_1 + \dots + k_{jn}x_n) - \frac{1}{\beta} S_n(p_1, \dots, p_n) \right\}$$

- Goal: entropical geometry of graph hypersurfaces of QFT

Work in progress: lexicographic semirings

- Min-plus type semirings widely used as “lexicographic semirings” in computational models of morphology and syntax in Linguistics
- Goal: use entropy deformations of these linguistics models, introducing an inverse temperature parameter β in finite-state representations of n -gram models based on the tropical semiring (modelled on thermodynamic formalism in dynamical systems)
- ... coming soon!