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Heights of toric varieties, entropy and integration over polytopes

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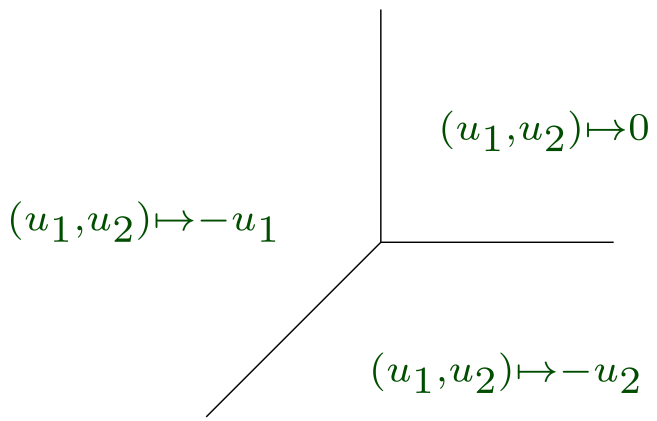
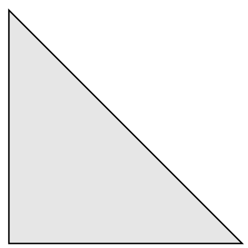
Toric varieties

Toric varieties form a remarkable class of algebraic varieties, endowed with an action of a torus having one Zariski dense open orbit. Toric divisors are those invariant by the action of the torus.

Together with their toric divisors, they can be described in terms of combinatorial objects such as lattice fans, support functions or lattice polytopes





Each cone corresponds to an affine toric variety and the fan encodes how they glue together. If the fan is complete then the toric variety is proper.

The support function determines a toric divisor D on each affine toric chart. By duality, the stability set of the support function is a polytope Δ, which may be empty but which is of dimension n as soon as D is nef, which is equivalent to the support function being concave.

One fundamental result is: if D is a toric nef divisor then

$$\deg_D(X) = n!\mathrm{vol}_n(\Delta).$$

Heights

A height measures the complexity of objects over the field of rational numbers, say. For $a/b \in \mathbf{Q}^\times$ and $d = \gcd(a, b)$:

$$\mathrm{h}(a/b) = \log\max(|a/d|, |b/d|) = \sum_v \log\max(|a|_v, |b|_v),$$

thanks to the product formula:

$$\prod_v |d|_v = 1$$

for any $d \in \mathbf{Q}^\times$ and where v runs over all the (normalised) absolute values on $\mathbf{Q}$ (usual and p-adic).

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For points of a projective space $x = (x_0 : \ldots : x_N) \in \mathbf{P}_N(\mathbf{Q})$:

$$\mathrm{h}(x) = \sum_v \log\|x\|_v = -\sum_v \log\|\ell(x)\|_v,$$

where $\|\cdot\|_v$ is a norm on $\mathbf{Q}^{N+1}$ compatible with the absolute value $|\cdot|_v$ on $\mathbf{Q}$ (usual or p-adic). Metrics on $\mathcal{O}_{\mathbf{P}_N}(1)$: $\|\ell(x)\|_v = \frac{|\ell(x)|_v}{\|x\|_v}$.

On an abstract variety equipped with a divisor (X, D), defined over $\mathbf{Q}$, the suitable arithmetic setting amounts to a collection of metrics on the space of rational sections of the divisor, compatible with the absolute values on $\mathbf{Q}$ (the collection is in bijection with the set of absolute values on $\mathbf{Q}$). We denote $\overline{D}$ the resulting metrised divisor.

Arithmetic intersection theory allows to define the height of X relative to $\overline{D}$ analogously to the degree $\deg_D(X)$:

$$\mathrm{h}_{\overline{D}}(X) = \sum_v \mathrm{h}_v(X)$$

where the local heights h_v are defined through an arithmetic analogue of Bézout formula. Local heights depend on the choice of auxiliary sections but the global height does not.

Metrics on toric varieties

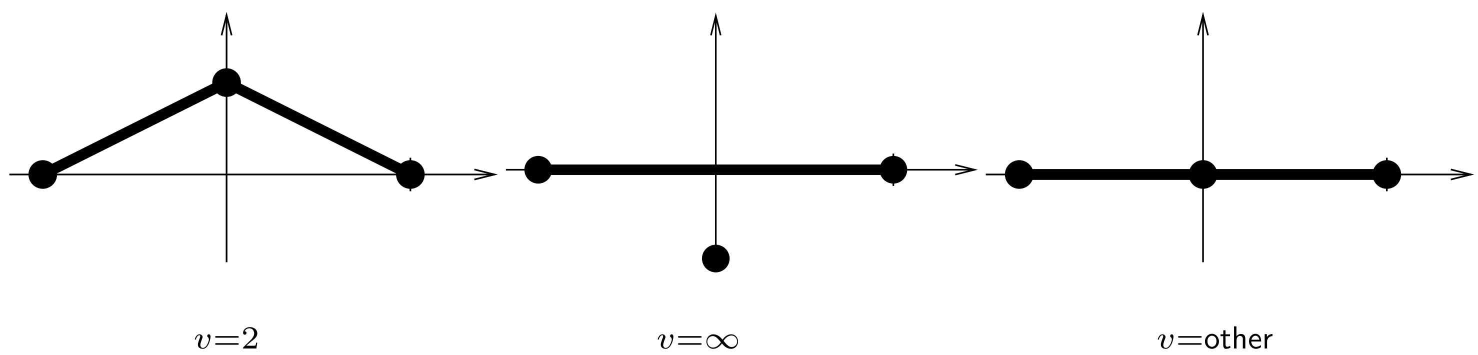
On toric divisors, a metric is said toric if it is invariant by the action of the compact sub-torus of the principal orbit.

There exists a bijection between toric metrics and continuous functions on the fan, whose difference with the support function is bounded. The metric is semipositive iff the corresponding function is concave.

By Legendre duality, the semipositive toric metrics are also in bijection with the continuous, concave functions on the polytope associated to the toric divisor, dubbed roof function.

The roof function is the concave enveloppe of the graph of the function $s \mapsto -\log \|s\|_{v,\sup}$, for s running over the toric sections of the divisor and its multiples.

Roof function of the pull-back of the canonical metric of $\mathbf{P}_2$ on $\mathbf{P}_1$ by $t \mapsto (\frac{1}{t}:\frac{1}{2}:t)$



The support function itself corresponds to the so-called canonical metric. Its roof function is the zero function on the polytope.

Heights on toric varieties

Let $(X, \overline{D})$ be a toric varieties with a toric divisor (over $\mathbf{Q}$), equipped with a collection of toric metrics (a toric metrised divisor).

The (local) roof functions attached to the toric metrised divisor sum up in the so-called global roof function:

$$\vartheta := \sum_v \vartheta_v.$$

We have the analogue of the formula seen for the degree:

$$\mathrm{h}_{\overline{D}}(X) = (n+1)! \int_\Delta \vartheta.$$

Metrics from polytopes

Let $\ell_F(x) = \langle x, u_F\rangle + \ell_F(0)$ be the linear forms defining a polytope $\Gamma \subset \mathbf{R}^n$, with F running over its facets and $\|u_F\| = \frac{\mathrm{vol}_{n-1}(F)}{n\mathrm{vol}_n(\Gamma)}$. Let $\Delta \subset \Gamma$ be another polytope, the restriction of

$$\vartheta := -\frac{1}{c}\sum_F \ell_F \log(\ell_F)$$

to Δ, is the roof function of some (archimedean) metric on the toric variety X and divisor D defined by Δ, hence $\overline{D}$.

Example: the roof function of the Fubini-Study metric on $\mathbf{P}_n$ is

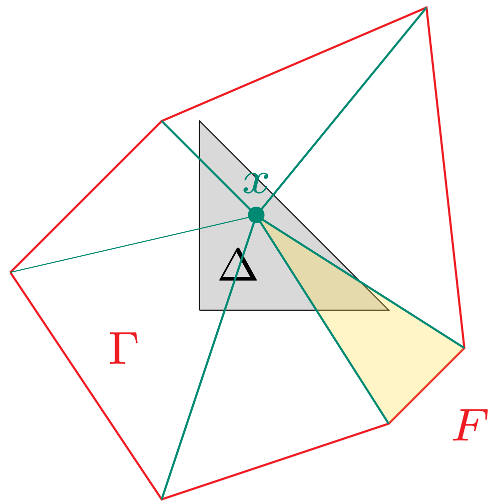
$$-(1/2)(x_0 \log(x_0) + \ldots + x_n \log(x_n))$$

where $x_0 = 1 - x_1 - \ldots - x_n$ (dual to $-\frac{1}{2}\log\left(1 + \sum_{i=1}^n \mathrm{e}^{-2u_i}\right)$).

Height as average entropy

Let $x \in \Gamma$ and β_x be the (discrete) random variable that maps $y \in \Gamma$ to the face F of Γ such that $y \in \mathrm{Cone}(x, F)$:

$$P(\beta_x = F) = \mathrm{dist}(x, F)\frac{\mathrm{vol}_{n-1}(F)}{n\mathrm{vol}_n(\Gamma)}.$$



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The entropy

$$\mathcal{E}(\beta_x) = -\sum_F P(\beta_x = F)\log(P(\beta_x = F))$$

satisfies

$$\frac{1}{\mathrm{vol}_n(\Delta)} \cdot \int_\Delta \mathcal{E}(\beta_x)\mathrm{dvol}_n(x) = \frac{c}{n+1} \cdot \frac{\mathrm{h}_{\overline{D}}(X)}{\deg_D(X)}.$$

Integration over polytopes

An aggregate of Δ in a direction $u \in \mathbf{R}^n$ is the union of all the faces of Δ contained in $\{x \in \mathbf{R}^n \mid \langle x, u\rangle = \lambda\}$ for some $\lambda \in \mathbf{R}$.

Definition – Let V be an aggregate in the direction of $u \in \mathbf{R}^n$, we set recursively: If $u = 0$, then $C_n(\Delta, 0, V) = \mathrm{vol}_n(V)$ and $C_k(\Delta, 0, V) = 0$ for $k \neq n$. If $u \neq 0$, then

$$C_k(\Delta, u, V) = -\sum_F \frac{\langle u_F, u\rangle}{\|u\|^2} C_k(F, \pi_F(u), V \cap F),$$

where the sum is over the facets F of Δ. This recursive formula implies that $C_k(\Delta, u, V) = 0$ for all $k > \dim(V)$.

Proposition [2, Prop.6.1.4] – Let $\Delta \subset \mathbf{R}^n$ be a polytope of dimension n and $u \in \mathbf{R}^n$. Then, for any $f \in \mathcal{C}^n(\mathbf{R})$,

$$\int_\Delta f^{(n)}(\langle x, u\rangle)\mathrm{dvol}_n(x) = \sum_{V \in \Delta(u)} \sum_{k=0}^{\dim(V)} C_k(\Delta, u, V) f^{(k)}(\langle V, u\rangle).$$

The coefficients $C_k(\Delta, u, V)$ are determined by this identity.

Example: If $\Delta = \mathrm{Conv}(\nu_0, \ldots, \nu_n) = \bigcap_{i=0}^n \{x; \langle x, u_i\rangle \geq \lambda_i\}$ is a simplex and $u \in \mathbf{R}^n \setminus \{0\}$, then $C_0(\Delta, u, \nu_0)$ equals

$$\frac{n!\mathrm{vol}_n(\Delta)}{\prod_{i=1}^n \langle \nu_0 - \nu_i, u\rangle} = \frac{\varepsilon \det(u_1, \ldots, u_n)^{n-1}}{\prod_{i=1}^n \det(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n)},$$

with ε the sign of $(-1)^n \det(u_1, \ldots, u_n)$.

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Ouf!