Matrix realization of a homogeneous cone

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§1. Introduction

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V : real vector space

Ω : regular open convex cone in V, that is,

- $V \supset \Omega$: open subset,
- $x \in \Omega,\ c > 0 \Rightarrow cx \in \Omega$,
- $x, y \in \Omega,\ 0 \leq t \leq 1 \Rightarrow (1-t)x + ty \in \Omega$,
- $\overline{\Omega} \cap (-\overline{\Omega}) = \{0\}$.

Ω : homogeneous cone

if $\exists G$: Lie group acting on Ω transitively as linear transforms

Example 1.

$V = \mathrm{Sym}(n, \mathbb{R})$

$\Omega = \mathcal{P}_n := \{ X \in \mathrm{Sym}(n, \mathbb{R}) \mid X \text{ is positive definite} \}$

$G = GL(n, \mathbb{R})$ acts on $\mathcal{P}_n$ transitively by

$\rho(A)X := AX\,{}^{\mathrm{t}}\!A \quad (A \in GL(n, \mathbb{R}),\ X \in \mathcal{P}_n)$.

$H_n := \left\{ T \in GL(n, \mathbb{R}) \mid T_{ij} = 0\ (i < j), \quad T_{ii} > 0\ (i = 1, \dots, n) \right\}$

H_n acts on $\mathcal{P}_n$ simply transitively by ρ

because of the Cholesky decomposition:

$\forall X \in \mathcal{P}_n\ \exists_1 T \in H_n$ s.t. $X = T\,{}^{\mathrm{t}}T$.

Example 2.

$$V := \{ X \in \mathrm{Sym}(3,\mathbb{R}) \mid X_{12} = X_{21} = 0 \}$$

$$= \left\{ X = \begin{pmatrix} x_1 & 0 & x_4 \\ 0 & x_2 & x_5 \\ x_4 & x_5 & x_3 \end{pmatrix} \mid x_1, \ldots, x_5 \in \mathbb{R} \right\}$$

$$\Omega := V \cap \mathcal{P}_3$$

$$H := \left\{ T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ t_4 & t_5 & t_3 \end{pmatrix} \mid t_1, t_2, t_3 > 0,\ t_4, t_5 \in \mathbb{R} \right\} \subset H_3$$

Then H acts on Ω simply transitively by ρ.

Example 3.

$n \geq 3$

$$V := \left\{ X = \begin{pmatrix} x_1 & & & x_3 \\ & \ddots & & \vdots \\ & & x_1 & x_n \\ x_3 & \dots & x_n & x_2 \end{pmatrix} \,\middle|\, x_1, \dots, x_n \in \mathbb{R} \right\}$$

$$\Omega := \mathcal{Z} \cap \mathcal{P}_{n-1} = \left\{ X \,\middle|\, x_1 > 0,\ x_1 x_2 - x_3^2 - \cdots - x_n^2 > 0 \right\}$$

$$H := \left\{ T = \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_1 & \\ t_3 & \dots & t_n & t_2 \end{pmatrix} \,\middle|\, t_1, t_2 > 0,\ t_3, \dots, t_n \in \mathbb{R} \right\}$$

H acts on Ω simply transitively.

The cone $\Omega \subset \mathcal{P}_{n-1}$ is linearly isomorphic to the circular cone $\left\{ y \in \mathbb{R}^n \,\middle|\, y_1 > \sqrt{y_2^2 + \cdots + y_n^2} \right\}$ in $\mathbb{R}^n$

because $\begin{pmatrix} y_1 - y_2 & & & y_3 \\ & \ddots & & \vdots \\ & & y_1 - y_2 & y_n \\ y_3 & \cdots & y_n & y_1 + y_2 \end{pmatrix} \in \Omega$

iff $y_1 > \sqrt{y_2^2 + \cdots + y_n^2}$.

Roughly speaking, every homogeneous cone is realized similarly.

§2. Matrix realization and left-symmetric algebra

Put $\mathfrak{h}_n := \left\{ T \in \mathrm{Mat}(n, \mathbb{R}) \,|\, T_{ij} = 0 \ (i < j) \right\} = \mathrm{Lie}(H_n)$.
For $X \in \mathrm{Sym}(n, \mathbb{R})$, define $\underset{\vee}{X} \in \mathfrak{h}_n$ by

$$(\underset{\vee}{X})_{ij} := \begin{cases} X_{ij} & (i > j) \\ X_{ii}/2 & (i = j) \\ 0 & (i < j) \end{cases}$$

Then $X = \underset{\vee}{X} + {}^{\mathrm{t}}(\underset{\vee}{X})$.

For $X, Y \in \mathrm{Sym}(n, \mathbb{R})$, define
$X \triangle Y := \underset{\vee}{X} Y + Y {}^{\mathrm{t}}(\underset{\vee}{X}) \in \mathrm{Sym}(N, \mathbb{R})$.
Then $\triangle$ gives a bilinear product on the vector space $\mathrm{Sym}(n, \mathbb{R})$, encoding the action ρ of H_n on $\mathcal{P}_n$.

Main Theorem. (i) Let $\mathcal{Z}$ be a subspace of Sym$(n, \mathbb{R})$ such that $\mathcal{Z} \triangle \mathcal{Z} \subset \mathcal{Z}$ and $E_n \in \mathcal{Z}$. Then $\mathcal{P}_{\mathcal{Z}} := \mathcal{Z} \cap \mathcal{P}_n$ is a homogeneous cone. The set $H_{\mathcal{Z}} := H_n \cap \left\{ \underset{\vee}{X} \mid X \in \mathcal{Z} \right\}$ forms a subgroup of H_n and acts simply transitively on $\mathcal{P}_{\mathcal{Z}}$.

(ii) Every homogeneous cone is linearly isomorphic to such $\mathcal{P}_{\mathcal{Z}}$.

Examples 2 and 3 are special cases.

(Recall $\mathcal{Z} = \left\{ \begin{pmatrix} x_1 & 0 & x_4 \\ 0 & x_2 & x_5 \\ x_4 & x_5 & x_3 \end{pmatrix} \mid x_1, \ldots, x_5 \in \mathbb{R} \right\}$

in Example 2).

The algebra $(\text{Sym}(n,\mathbb{R}),\triangle)$ has the following properties:

(C1) $X\triangle(Y\triangle Z)-(X\triangle Y)\triangle Z = Y\triangle(X\triangle Z)-(Y\triangle X)\triangle Z$ for all X,Y,Z (*left-symmetry*)

(C2) there exists a linear form ξ such that $\xi(X\triangle X) > 0$ for all non-zero X (*compactness*)

(C3) For each X, the left-multiplication operator $L_X : Y \mapsto X\triangle Y$ has only real eigenvalues (*normality*)

(C4) $E_n\triangle X = X\triangle E_n = X$ for all X ($\exists$*unit element*).

An $\mathbb{R}$-algebra $(V,\triangle)$ satisfying (C1) is called a left-symmetric algebra (or Koszul-Vinberg algebra), while a left-symmetric algebra satisfying (C2) and (C3) is called a clan (a compact normal left-symmetric algebra).

Vinberg obtained a one-to-one correspondence between a homogeneous cone and a clan with unit element up to natural isomorphisms.

Theorem. Every clan with a unit element is isomorphic to a subalgebra of $(\mathrm{Sym}, \triangle)$.

Theorem. A subalgebra $\mathcal{Z}$ of $(\mathrm{Sym}, \triangle)$ with $E_n \in \mathcal{Z}$ admits a specific block decomposition after an appropriate permutation of rows and columns (see Proceedings).

§3. Homogeneous Hessian metrics

For $X \in \mathrm{Sym}(n,\mathbb{R})$ and $k = 1,\dots,n$,
let $X^{[k]} := (X_{ij})_{1\le i,j\le k} \in \mathrm{Sym}(k,\mathbb{R})$.
For $X \in \mathcal{P}_n$ and $\underline{s} = (s_1,\dots,s_n) \in \mathbb{R}^n_{>0}$, define
$\Delta_{\underline{s}}(X) := \prod_{k=1}^n (\det X^{[k]})^{s_k - s_{k+1}}$, where $s_{n+1} := 0$.
If $X = T\,{}^{\mathrm{t}}T$ with $T \in H_n$, then $\Delta_{\underline{s}}(X) = \prod_{k=1}^n (T_{kk})^{2s_k}$
Therefore,
$\Delta_{\underline{s}}(\rho(T)Y) = (\prod_{k=1}^n (T_{kk})^{2s_k})\Delta_{\underline{s}}(Y) \quad (Y \in \mathcal{P}_n)$.

Let $g_{\underline{s}}$ be the Hessian metric on $\mathcal{P}_n$ whose potential is $-\log \Delta_{\underline{s}}(Y)$. Then $g_{\underline{s}}$ is H_n-invariant.
For $X \in \mathcal{P}_n$ and $A, B \in T_X\mathcal{P}_n \equiv \mathrm{Sym}(n,\mathbb{R})$, we have
$g_{\underline{s}}(A,B)_X := \sum_{k=1}^n (s_k - s_{k+1}) \mathrm{Tr}\left(A^{[k]}(X^{[k]})^{-1}B^{[k]}(X^{[k]})^{-1}\right)$.

A Hessian metric g on a domain $D \subset \mathbb{R}^n$ is said to be homogeneous if $\exists G$: Lie group acting on D transitively as affine isometries.

Clearly the Hessian metric $g_{\underline{s}}$ on $\mathcal{P}_n$ is homogeneous. Moreover, Every homogeneous Hessian metric on $\mathcal{P}_n$ is equivalent to some $g_{\underline{s}}$. Namely, for a homogeneous Hessian metric g on $\mathcal{P}_n$, there exists a linear transform $f : \mathcal{P}_n \to \mathcal{P}_n$ such that $g = f^* g_{\underline{s}}$.

Theorem. Let $\mathcal{Z}$ be a subalgebra of $(\mathrm{Sym}(n, \mathbb{R}), \triangle)$ with $E_n \in \mathcal{Z}$. Then every homogeneous Hessian metric on the homogeneous cone $\mathcal{P}_{\mathcal{Z}}$ is equivalent to the restriction $g_{\underline{s}}|_{\mathcal{P}_{\mathcal{Z}}}$ with some $\underline{s} \in \mathbb{R}^n_{>0}$.

This parametrization is redundant because different $\underline{s}$ may give the same metric on $\mathcal{P}_{\mathcal{Z}}$.
See Proceedings for a precise parametrization.