Multiply CR- Warped Product Statistical Submanifolds of a Holomorphic Statistical Space From

By

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CR-submanifold and CR-warped Product Submanifold

Let *B* and *F* be two Riemannian Manifolds with Riemannian metric g_B and g_F, respectively , and *f* a positive differentiable function on *B*. The warped product manifold $B \times F$ equipped with the Riemannian metric

$$g = g_B + f^2 g_F$$

The function f is called the warping function. It is well known that the notion of warped product plays some important roles id differential geometry as well as in physics.

Let *M* be a Kaehler manifold with complex structure *J* and *N* a Riemannian manifold isometrically immersed in $\overline{M}$. For each $x \in N$, we denote by D_x the maximal holomorphic subspace of the tangent space $T_x N$ of *N*. If the dimension of D_x is the same for all $x \in N$, the space D_x define a holomorphic distribution *D* on *N,* which is called the holomorphic distribution of *N*. A submanifold N is a Kaehler manifold $\overline{M}$ is called a CR-submanifold if there exists a holomorphic distribution D on N whose orthogonal complement $D^{\perp}$ is totally real distribution, i.e., $JD^{\perp} \subset T^{\perp} N$. A CR-submanifold is called a totally real submanifold if dim D_x=0.

Statistical manifolds introduced, in 1985, by Amari have been studied in term of information geometry. Since the geometry of such manifolds includes the notion of dual connections, also called conjugate connection in affine geometry, it is closely related to the affine differential geometry. Further, a statistical structure being a generalization of a Hessian geometry.

Let $(\overline{M}, \overline{g})$ be Riemannian manifold and M a submanifold of $\overline{M}$. If (M,∇,g) is a statistical manifold, then we call (M,∇,g) a statistical submanifold of $(\overline{M}, \overline{g})$, where ∇ is an affine connection on *M* and *g* is the metric tensor on *M* induced from the Riemannian metric $\overline{g}$ on $\overline{M}$. Let $\overline{\nabla}$ be an affine connection on $\overline{M}$. If $(\overline{M}, \overline{g}, \overline{\nabla})$ is astatistical manifold and *M* a submanifold of $\overline{M}$, then (M,∇,g) is also a statistical manifold by induced connection ∇ and metric *g*.

In the case $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold, the induced metric connection *g* has to be non-degenerated.

In the geometry of submanifolds, Gauss formula, Weingarten formula and the equation of Gauss, Codazzi and Ricci are know as fundamental equations. Corresponding fundamental equations on statistical submanifolds were obtained .

Let *M* be an n-dimensional submanifold of $\overline{M}$. Then, for any $X, Y \in \Gamma(TM)$, Gauss formula is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

$$\overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y)$$

Where *h* and h^* are symmetric and bilinear, called the imbedding curvature tensor of *M* in $\overline{M}$ for $\overline{\nabla}$ and the imbedding curvature tensor of *M* in $\overline{M}$ for $\overline{\nabla}^*$, respectively.

it is also proved that (∇, g) and (∇^*, g) are dual statistical structure on *M*, where *g* is induced metric on $\Gamma(TM)$ from the Riemannian metric $\overline{g}$ on $\overline{M}$.

Let us denote the normal bundle on *M* by $\Gamma(TM^{\perp})$. Since h and h^* are bilinear, we have the linear transformation A_{ξ} and A^*_{ξ} defined by

$$g(A_{\xi}X, Y) = \overline{g}(h(X, Y), \xi)$$

$$g(A^*_{\xi}X, Y) = \overline{g}(h^*(X, Y), \xi)$$

Definition. Let $N_1, N_2, ..., N_k$ be Riemannian manifold of the dimensions $n_1, n_2, ..., n_k$ respectively and let $N = N_1, N_2, ..., N_k$ be the Cartesian product of $N_1, N_2, ..., N_k$. For each a, denote by $\pi_a : N \to N_a$ the canonical projection *N* and N_a . We denote the horizontal lift of N_a in N via π_a by N_a itself. If $\sigma_2, ...\sigma_k : N_1 \to R^+$ are positive valued functions, then

$$(2.1) \qquad g(X,Y) = \langle \pi_{1*}X, \pi_{1*}Y \rangle + \sum_{a=1}^{k} (\sigma_a \circ \pi_1)^2 \langle \pi_{a*}X, \pi_{a*}Y \rangle$$

define a metric *g* on *N*. The product manifold *N* endowed with this metric is denoted by $N_1 \times_{\sigma 2} N_2 \times .... \times_{\sigma k} N_k$. This product manifold *N* is known as multiply warped product manifold.

Definition. If $N_1, N_2, \ldots., N_k$ be k statistical manifolds, then *N*= $N_1 \times_{\sigma 2} N_2 \times \ldots. \times_{\sigma k} N_k$ is again a statistical manifold with metric given by equation (2.1). This manifold *N* is called multiply warped product statistical manifold.

Now let us denote the part $\sigma_2 N_2 \times \ldots \times \sigma_k N_k$ by $N_\perp$ and N_1 by N_T. Then *N* can be represented as $N = N_T \times N_\perp$. We denote by $\overline{X}, \overline{Y} \ldots \in \Gamma(M)$ as the vector field on *M* and *X, Y*.... the induced vector field on *N*.

Definition. A multiply warped product statistical submanifold $N = N_T \times N_\perp$ in an almost complex manifold *M* is called a multiply CR-warped product statistical submanifold if N_T is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold of *M*.

We denote by R^{2m}, $m \geq 1$ the Euclidean 2m space with the standard metric. Then the canonical complex structure of R^{2m} is defined by

$$J(x_1, y_1, \ldots, x_m, y_m) = (-y_1, x_1, \ldots, -y_m, x_m) \qquad (*)$$

Example. Consider in R^8 the submanifold is given by the equations [B. Sahin, Geom. Dedicata 2006]

$$x_1 = t\cos\theta,\ x_2 = s\cos\theta,\ x_3 = t\cos\varphi,\ x_4 = s\cos\varphi;$$

$$x_5 = t\sin\theta,\ x_6 = s\sin\theta,\ x_7 = t\sin\varphi,\ x_8 = s\sin\varphi, \qquad \theta, \varphi \in (0, \frac{\pi}{2})$$

From (*) one can obtain that TM is spanned by Z_t, Z_s, Z_θ, Z_φ , where

$$Z_t = cos\theta\, \partial x_1 + cos\varphi\, \partial x_3 + sin\theta\, \partial x_5 + sin\varphi\, \partial x_7,$$

$$Z_s = cos\theta\, \partial x_2 + cos\varphi\, \partial x_4 + sin\theta\, \partial x_6 + sin\varphi\, \partial x_8,$$

$$Z_\theta = -t\, sin\theta\, \partial x_1 - s\, sin\theta\, \partial x_2 + t\, cos\theta\, \partial x_5 + s\, cos\theta\, \partial x_6$$

$$Z_\varphi = -t\, sin\varphi\, \partial x_3 - s\, sin\varphi\, \partial x_4 + t\, cos\varphi\varphi\, \partial x_7 + s\, cos\varphi\, \partial x_8$$

Using (*) one gets that $D = span\{Z_t , Z_s\}$ is invariant with respect to *J*. Moreover, $JZ_\theta \text{ and } JZ_\varphi$ are orthogonal to *TM*. Hence, $D^\perp = span\{Z_\theta , Z_\varphi\}$ is anti-invariant with respect to *J*. Thus *M* is a CR-submanifold of R^8. Furthermore, we can derive that D and $D^\perp$ are integralable.

Denoting the integral manifold of D and $D^{\perp}$ by M_T *and* $M_{\perp}$, respectively, then the induced metric tensor is

$$g = 2dt^2 + 2ds^2 + (t^2 + s^2)(d\theta^2 + d\varphi^2)$$

$$= g_{M_T} + (t^2 + s^2) g_{M^{\perp}}$$

Thus M is a CR-warped product submanifold of R^8 with warping function $f = \sqrt{t^2 + s^2}$.

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From the decomposition of $TN = D \oplus D^{\perp}$ and $T^{\perp}N = JD^{\perp} \oplus \lambda$ we may write

$$h(X,Y) = h_{JD^{\perp}}(X,Y) + h_{\lambda}(x,y)$$

Also for multiply CR- warped product statistical submanifolds *N* of a statistical manifold [L. Tod., Diff. Geom. – Dynamical system ,2006]

$$z = \sum_{a=2}^{k} (X(\log \sigma_a))z^a \quad \text{and} \quad \nabla^*_X Z = \sum_{a=2}^{k} (X(\log \sigma_a))Z^a \qquad (4)$$

for any vector fields $X \in D$ and $Z \in D^{\perp}$, where Z^a denotes the N_a- component of *Z* .

Lemma 1. *Let* $N = N_T \times_{\sigma_2} N_2 \times \ldots\ldots\ldots\ldots\ldots\ldots \times_{\sigma_k} N_k$ *be a multiply CR-warped product statistical submanifold of a holomorphic statistical space form M. then we have*

(i) $$h_{JD^{\perp}}(JX, Y) = \sum_{a=2}^{k} (X(\log \sigma_a))JZ^a + JP_z JX$$

(ii) $$g(P_Z JX, W) = g(Q_Z JX, JW)$$

(iii) $$g(h(JX, Z), Jh(X, Z)) = \|h_\lambda(Z, X)\|^2 + g(Q_Z X, Jh_\lambda(X, Z))$$

For any vector field X in D and Z , W in $D^{\perp}$ *, where* Z^a *denotes the* N_a *- component of Z.*

Proof. From Gauss formula we can write

$$\nabla_Z JX + h(JX, Y) = P_Z X + Q_Z X + J\nabla_Z X + Jh(X, Z)$$

$$h(JX, Z) = P_Z X + Q_Z X + J(\sum_{a=2}^{k}(X(\log\sigma_a)Z^a) + Jh(Z, X)$$

$$-\sum_{a=2}^{k}(JX(\log\sigma_a))JZ^a \qquad (5)$$

where *P* and *Q* denotes the tangential and normal projection.
Comparing the tangential part in the above equation and then taking inner product with $W \in D^{\perp}$, we get

$$h_{JD^{\perp}}(JX, Z) = \sum_{a=2}^{k}(X(\log\sigma_a))Z^a + JP_Z JX, \forall X \in D, Z \in D^{\perp}$$

Now comparing normal parts of (5) and taking inner product with *JW for* $W \in D^{\perp}$

$$g(h_{JD^{\perp}}(JX, Z), JW)g(Q_Z X, JW) + \sum_{a=2}^{k}(X(\log\sigma_a)g(JZ^a, JW))$$

Using part(i) of the **lemma 1** we arrive at

$$g(P_Z JX, W) = g(Q_Z X, JW).$$

Comparing normal part of $h(JX,Z)-Jh_{\lambda}(Z,X)=Q_Z X+\sum_{a=2}^{k}(X(\log\sigma_a))JZ^a$

on both the sides and taking inner product with $Jh(X,Z)$ we find

$$g(h(JX,Z),Jh(X,Z)) = \|h_{\lambda}(Z,X)\|^2 + g(Q_Z X, Jh_{\lambda}(X,Z))$$

Theorem 2. *Let* $N = N_T \times_{\sigma_2} N_2 \times \dots\dots\dots\dots\dots\dots \times_{\sigma_k} N_k$ *be multiply CR-warped product statistical submanifold of holomorphic statistical space form M with* $P_{D^{\perp}}D \in D$ *, then the square norm of imbedding curvature tensor of N in M satisfies the following inequalities :*

$$\|h\|^2 \geq \sum_{a=2}^{k} n_a{}^2 \ \|\nabla \log\sigma_a\|^2 + \|P_{D^{\perp}} D\|^2$$

Proof. Let $\{X_1, X_2, \ldots, X_p, X_{p+1} = JX_1, \ldots, JX_{2p} = JX_p\}$ be local orthonormal frame of vector field frame of the vector field on N_T and $\{Z_1, Z_2, \ldots, Z_q\}$ be such that Z_{Δ_a} is a basis for some N_a,

a= 2,.....,k where

$$\Delta_2 = \{1,2,\ldots, n_2\}, \ldots, \Delta_k = \{n_2 + n_3 + \ldots + n_{k-1} + 1, \ldots, n_1 + n_2 + \ldots + n_k\}$$

and $$n_2 + n_3 + \ldots + n_k = q$$

$$\|h\|^2 = \sum_{i,j=1}^{2P} g\left(h(X_i, X_j), h(X_i, X_j)\right)$$

$$+ \sum_{i=1}^{2P} \sum_{a=2}^{k} g\left(h(X_i, Z_{\Delta_a}), h(X_i, Z_{\Delta_a})\right)$$

$$+ \sum_{a,b=2}^{k} g\left(h(Z_{\Delta_a}, Z_{\Delta_b}), h(Z_{\Delta_a}, Z_{\Delta_b})\right)$$

The above equation implies

$$\|h\|^2 \geq \sum_{i=1}^{2P}\sum_{a=2}^{k} g\left(h(X_i, Z_{\Delta_a}), h(X_i, Z_{\Delta_a})\right)$$

Now using part (i) of the Lemma 1 we get

$$\|h\|^2 \geq \sum_{i=1}^{2P}\sum_{a=2}^{k} g\left(n_a(JX_i,(log\sigma_a))JZ_{\Delta_a} + JP_{Z_{\Delta_a}}X_i\,, n_a(JX_i,(log\sigma_a))JZ_{\Delta_a} + JP_{Z_{\Delta_a}}X_i\right)$$

In the view of the above assumption $P_{D^{\perp}}D \in D$, the above inequality takes the form

$$\|h\|^2 \geq \sum_{a=2}^{k} {n_a}^2 \ \|\nabla log\sigma_a\|^2 + \|\mathrm{P_{D^{\cdot}}}\ D\|^2$$

By the Cuachy-Schwartz inequality the above equation becomes

$$\sum_{a=2}^{k} n_a{}^2 \ \|\nabla log\sigma_a\|^2 \, \|Z_{\Delta_a}\|^2 + \ \|P_{D^\perp} D\|^2$$

$$\geq \sum_{a=2}^{k} n_a{}^2 \ \|(\nabla log\sigma_a) Z_{\Delta_a}\|^2 + \ \|P_{D^\perp} D\|^2$$

Therefore

$$\|h\|^2 \geq \sum_{a=2}^{k} n_a{}^2 \ \|\nabla log\sigma_a\|^2 + \ \|P_{D^\perp} D\|^2$$

Theorem 3. *Let* $N = N_T \times_{\sigma_2} N_2 \times \ldots\ldots\ldots\ldots\ldots\ldots \times_{\sigma_k} N_k$ *be a compact orientable multiply CR- warped product statistical submanifold without boundary of holomorphic statistical space form M of constant curvature k. If* $P_{D^\perp} D \in D$ *and* $A_{\nabla^\perp_X \cdot JZ} JX = A_{\nabla^\perp_{JX} \cdot JZ} X$

Then

$$K \leq 0$$

And the equality holds if and if $grad_D(\log\sigma_a) = 0$

Proof. Let $X \in D$, $Z \in D^\perp$, then form holomorphic statistical space form of constant curvature k, we have

$$\bar{R}(X, JX, Z, JZ)$$
$$= \frac{k}{4}\{g(JX, Z)g(X, JZ) - g(X, Z)g(JX, JZ)$$
$$- g(X, Z)g(JX, JZ) + g(JX, Z)g(X, JZ)$$
$$- 2g(X, X)g(JZ, JZ)\}$$

Which implies

$$\bar{R}(X,JX,Z,JZ) = \frac{-k}{2}\{g(X,X)g(Z,Z)\}$$
$$= \frac{-k}{2}\,\|X\|^2 \sum_{a=2}^{k} \|Z^a\|^2. \qquad (7)$$

On the other hand from Codazzi equation, we may write

$$\bar{R}(X,JX,Z,JZ)$$
$$= g(\nabla^{\perp}_X h(JX,Z),JZ) - g(h(\nabla_X JX,Z),JZ)$$
$$- g(h(JX,\nabla_X Z),JZ) - g(\nabla^{\perp}_{JX} h(X,Z),JZ)$$
$$+ g(h(\nabla_{JX} X,Z),JZ) + g(h(X,\nabla_{JX} Z),JZ) \qquad (8)$$

Now, we calculate each term of (8) as

$$g(\nabla^{\perp}_X h(JX,Z),JZ) = \sum_{a=2}^{k} [\{X(X\, log\sigma_a) + 2(X\, log\sigma_a)^2\} g(Z^a,Z^a) - g(h(JX,Z),\overline{\nabla}^*_X JZ)] \quad (10)$$

Similarly we replace X By JX in the last equation, we get

$$g\big(\nabla^{\perp}_{JX} h(X,Z)\big),JZ) = \sum_{a=2}^{k} \big[\{JX(JX\, log\sigma_a) + 2(JX\, log\sigma_a)^2\} g(Z^a,Z^a) - g(h(X,Z),\overline{\nabla}^*_{JX} JZ)\big] \quad (11)$$

$$g(h(JX,\nabla_X Z),JZ) = \sum_{a=2}^{k}(X\, log\sigma_a)^2\, g(Z^a,Z^a) \qquad (12)$$

$$g\big(h\big(X,\nabla_{JX} Z\big),JZ\big) = -\sum_{a=2}^{k}(JX\, log\sigma_a)^2\, g(Z^a,Z^a) \qquad (13)$$

$$g\big(h\big(\nabla_{JX} X,Z\big),JZ\big) = -\sum_{a=2}^{k}(J\nabla_{JX} X\, log\sigma_a)\, g(Z^a,Z^a) \qquad (14)$$

$$g\big(h\big(\nabla_{JX} X,Z\big),JZ\big) = -\sum_{a=2}^{k}(J\nabla_X JX\, log\sigma_a)\, g(Z^a,Z^a)$$

$$\overline{R}(X,JX,Z,JZ)=\sum_{a=2}^{k}[\{X(X\log\sigma_a)+(X\log\sigma_a)^2\}g(Z^a,Z^a)]$$

$$-g(h(JX,Z),\overline{\nabla}^*_X JZ)+\sum_{a=2}^{k}[\{JX(JX\log\sigma_a)+(JX\log\sigma_a)^2\}g(Z^a,Z^a)]$$

$$-g(h(JX,Z),\overline{\nabla}^*_{JX}JZ)-\sum_{a=2}^{k}(\nabla_X X\log\sigma_a)g(Z^a,Z^a)$$

$$-\sum_{a=2}^{k}\{(\nabla_{JX}JX\log\sigma_a)g(Z^a,Z^a)\} \qquad (16)$$

Combining (7) and (16) and taking summation over the range from 1 to p, we have

$$\left(\frac{pk}{4}\right)\sum_{a=2}^{k}\left\|Z^{a}\right\|^{2}=\sum_{a=2}^{k}\Delta(\log\sigma_{a})\left\|Z^{a}\right\|^{2}$$

$$-\sum_{a=2}^{k}\left\|grad_{D}(\log\sigma_{a})\right\|^{2}$$

$$+\sum_{i=1}^{p}[g(h(Je_{i},Z),\nabla_{e_{i}}^{\perp *})JZ-g(h(e_{i},Z),\nabla_{Je_{i}}^{\perp *}JZ)] \quad (18)$$

Integrating both the sides, Green's and the hypothesis leads to

$$k = \frac{-4\sum_{a=2}^{k} \left\| Z^a \right\|^2 \int_N \{\left\| grad_D(\log \sigma_a) \right\|^2\} dv}{p\sum_{a=2}^{k} \left\| Z^a \right\|^2 \int_N dv} \leq 0$$

Since $$p\sum_{a=2}^{k} \left\| Z^a \right\|^2 \int_N dv > 0$$

And $$\sum_{a=2}^{k} \left\| Z^a \right\|^2 \int_N \{\left\| grad_D(\log \sigma_a) \right\|^2\} dv \geq 0$$

Further the equality holds if and only if $\int_N \{\left\| grad_D(\log \sigma_a) \right\|^2\} dv = 0$

Which implies that the equality holds if $grad_D(log\sigma_a) = 0$. This proves the theorem.

Theorem 4. *Let* $N = N_T \times_{\sigma_2} N_2 \times \dots\dots\dots\dots\dots\dots \times_{\sigma_k} N_k$ *be a compact orientable anti-invariant multiply warped product statistical submanifold without boundary of holomorphic statistical space form M of constant curvature k. If* $P_{D^\perp} D \in D$ *and* $A_{\nabla^{\perp}_{X^*} JZ} JZ = A_{\nabla_{JX^*} JX} X$*, then*

$$R(X, Y, X, Y) \geq g(H, H^*)$$

and the equality holds if and only if $grad_D(\log \sigma_a) = 0$

Proof. From the previous theorem we have $K \leq 0$. Since N is anti-invariant , we have $N_T = 0$ and $N = N_\perp$.

This implies that *N* becomes completely totally umbilical submanifold of *M*. Furthermore, from the expression of the ambient curvature we have, for two orthonormal vector $X, Y \in TN$

Then $$\overline{R}(X, Y, X, Y) = -\frac{k}{4}.$$

Furthermore, from Gauss equation and totally umbilicity of N, we obtain

$$R(X,Y,X,Y) = (-\frac{k}{4} + g(H,H*))$$

$$R(X,Y,X,Y) \geq g(H,H^{*})$$

and the equality holds if $grad_D(log\sigma_a) = 0$

Theorem 5. *Let* $N = N_T \times_{\sigma_2} N_2 \times \dots\dots\dots \times_{\sigma_k} N_k$ *be a compact orientable anti-invariant multiply warped product statistical submanifold without boundary of holomorphic statistical space form M of constant curvature k. If* $P_{D^\perp} D \in D$ *and* $A_{\nabla^\perp_{X*} JZ} JX = A_{\nabla^\perp_{JX^*} JZ} X$ *, then M is Einstein and N is Einstein if and only if*

$$\frac{k}{4} + g(H, H^*) \text{ is constant.}$$

Proof. The proof is straight from the last theorem and the Gauss equation which combinely give

$$Ric(Y,Z) = (n-1)\{\frac{k}{4} + g(H,H^*)\}g(Y,Z)$$

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Thank you for attention!