Vinberg's theory of homogeneous convex cones, Development and applications

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Transformation groups. Conference in honor of E.B.Vinberg,
Moscow, 18 December 2017

I. Differential geometry of convex cones and domains: Characteristic function and invariant measure, characteristic hypersurface, canonical invariant Hessian metric , duality

Let $\mathcal{V} \subset V = \simeq \mathbb{R}^n$ be a convex cone (without straight lines),

$$\mathcal{V}^* = \{\xi \in V^*, \xi(\mathcal{V}) > 0\} \subset \mathcal{V}^*$$

the dual (convex) cone and $G(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}) \subset GL(V)$ the group of automorphisms.

The cone is characterised by a strictly convex homogeneous of degree $-n$ function

$$\varphi(a) = \int_{\mathcal{V}^*} e^{-\xi(a)} d\xi, \ a \in \mathcal{V}$$

called Vinberg-Koszul characteristic function.

It is the density of $G(\mathcal{V})$-invariant measure $\mathrm{vol} = \varphi(x)dx$

$\mathcal{V}_1 = \{\varphi = 1\}$ is called the characteristic hypersurface. The function $\frac{1}{\varphi}$ goes to zero at the boundary $\partial\mathcal{V}$.

There are two natural Riemannian metrics in $\mathcal{V}$:
The canonical $G(\mathcal{V})$-invariant metric $g^{can} = \partial^2 \log \varphi(x)$
and cone metric $g^{con} = \partial^2 \varphi$.
They coincide on the hypersurface $\mathcal{V}_1$.

There is a canonical map

$$* : V \supset \mathcal{V} \to \mathcal{V}^* \subset V^*,\ x \mapsto x^* = -d \log \varphi.$$

s.t. $(Ax)^* = (A^t)^{-1} x^*$ for $A \in G(\mathcal{V})$. The stability subgroup $K \subset G(\mathcal{V})$ of a point $o \in \mathcal{V}$ is a maximal compact subgroup of $G(\mathcal{V})$,

Homogeneous convex cones and domains

Let $\mathcal{V} = G(\mathcal{V})/K \subset V$ be a homogeneous convex cone. Then the maximal triangular subgroup $B \subset G(\mathcal{V})$ acts simply transitively on $\mathcal{V}$ and $G(\mathcal{V}) = K \cdot B$.

Moreover , the dual map $* : \mathcal{V} \to \mathcal{V}^*$ is involutive ($*^2 = \mathrm{Id}$) and any hyperplane section of a homogeneous cone is tangent at the center of gravity x_0 to the level set $\varphi(x) = \varphi(x_0)$.

Relation between homogeneous convex domains $D \subset A^n$ and homogeneous convex cones :

Any affine homogeneous domain $D \subset \mathbb{R}^n$ is extended to a homogeneous convex cone

$$\mathcal{V}(D) = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n,\ t > 0, tx \in D\}$$

Conversely, any homogeneous convex domain is a section of a homogeneous convex cone.

Application to Information Geometry (IG).

History of IG. Fisher-Rao metric and Kulback-Leibler divergence

IG studies differential geometry of a smooth (statistical) manifold of probability measures M with application to statistics and probability.

1)The basic idea of IG goes back to R.A. Fisher (Fisher matrix) and K.R. Rao who realized (1945) that Fisher matrix is a Riemannian metric (Fisher-Rao metric)

$$g_{ij}(\xi) = E(\partial_{\xi^i} \log p(x,\xi)\partial_{\xi^j} \log p(x,\xi))$$

in a manifold of probability measures $M = \{p(x, \xi^i)\}$ where x is a random variable and ξ^i are local coordinates.

2) Generalising Cl. Shannon notion of informational entropy (1948), S. Kulback and R. Leibler (1951) defined relative entropy (or KL divergence) between two probability distributions $p(x), q(x)$ on probability space X as

$$D_{KL}(p, q) = \int_X p(x) \log \frac{p(x)}{q(x)} d\mu(x).$$

As a new mathematical discipline ,IG was developed by N. Chentsov (1970s) and S-I Amari (1980s).
IG studies manifolds of probability measures with a divergence.

Divergence is a non negative function $D :: M \times M \to \mathbb{R}$ which vanishes only on the diagonal $M^d \subset M \times M$ and s.t.

$$g_{ij} = -\partial_{x^i} \partial_{y^j} D(x, y)|_{x=y}.$$

defines a Riemannian metric g on $M = M^d$. (Fisher-Rao metric). Chentsov proved the uniqueness of the Fisher-Rao metric , its invariancy (with respect to coordinate change) and monotonic property.

The rigorous proof of uniqueness was given 45 years later by N. Ay, J, Jost, H. V. Le and L. Schwachhöfer (2016).

Statistic and Hessian manifold

Chentsov associated with the divergence $D(x, y)$ a torsion free connection ∇ s.t. $S = \nabla g$ is a symmetric 3-form. Then the conjugated connection ∇^* is also torsion free.

Such pair (g, ∇) is called a statistical manifold and Hessian manifold (also dually flat, also affine Käehler manifold) if ∇ is flat.

The divergence $D(x, y)$ may be constructed from (g, ∇) (Nihat Ay Shun-ichi Amari, 2016, Entropy)

A convex cone with canonical metric is a Hessian manifold and Vinberg and Koszul results about characteristic function admits interpretation for one of the most interesting case of IG ("exponential family")

"An exponential family is an ideal model to study the dually flat (=Hessian) structures and also statistical inference "(Amari)

The cone of positively defined matrices as a statistical manifold.

One of the most important example of statistical manifold is the manifolds of Gauss distributions

$$\{ \quad G(x)_{g,\mu} = \frac{1}{(2\pi)^{n/2}(\det g)^{1/2}} e^{-\frac{1}{2}g(x-\mu,x-\mu)} \quad \}$$

, which is identified with the characteristic hypersurface SL_{n+1}/SO_{n+1} of the cone $\mathcal{P}_{n+1}$ of positively defined matrices.
M. Loves MFc, M. Min-Oo, E. A. Ruh,2000.
F. Barbaresco, Frechet Metric Space, Homogeneous Siegel Domains and Radar Matrix Signal Processing, in F. Barbaresco, F. Nielsen, O.Schwander, Matrix Information Geometry, Indo-French Workshop, 2011.

Works by Frederic Barbaresco on application of Koszul-Vinberg characteristic function to thermodynamics and statistical physics

"Koszul Information Geometry and Souriau Geometric Temperature/Capacity of Lie Group Thermodynamics "(Entropy, 2014).

«The Francois Massieu 1869 idea to derive some mechanical and thermal properties of physical systems from "Characteristic Functions was developed by Gibbs and Duhem in thermodynamics and by Poincare in probability. This paper deals with generalization of this Characteristic Function concept by Jean-Louis Koszul in Mathematics and by Jean-Marie Souriau in Statistical Physics. The Koszul-Vinberg Characteristic Function (KVCF) on convex cones will be presented as cornerstone of "Information Geometry"theory, defining Koszul Entropy as Legendre transform of minus the logarithm of KVCF, and Fisher Information Metrics as hessian of these dual functions, invariant by their automorphisms. »

« In parallel, Souriau has extended the Characteristic Function in Statistical Physics looking for other kinds of invariances through co-adjoint action of a group on its momentum space, defining physical observables like energy, heat and momentum as pure geometrical objects. In covariant Souriau model, Gibbs equilibriums states are indexed by a geometric parameter, the Geometric (Planck) Temperature, with values in the Lie algebra of the dynamical Galileo/Poincare groups, interpreted as a space-time vector, giving to the metric tensor a null Lie derivative. Fisher Information metric appears as the opposite of the derivative of Mean "Moment map"by geometric temperature, equivalent to a Geometric Capacity or Specific Heat.
The Koszul-Vinberg Characteristic Function (KVCF) is a dense knot in important mathematical fields such as Hessian Geometry, Kählerian Geometry and Affine Differential Geometry. As essence of Information Geometry, this paper develops KVCF as a transverse concept in Thermodynamics, in Statistical Physics and in Probability.»

II. Classification of self-dual homogeneous convex cones. Self-dual homogeneous convex cones and compact Jordan algebras (CJA).

A convex cone $\mathcal{V}$ is self-dual if $g_{Eucl}\mathcal{V} = \mathcal{V}^*$ for some Euclidean metric $g_{Eucl} : V \to V^*$.

A Jordan algebra J

($a \cdot b = b \cdot a = T_a b, [aba^2] = 0$ where

$[abc] = (a \cdot b) \cdot c - a \cdot (b \cdot c)$)

is compact if $g(a, a) = \mathrm{tr}\ T_{a^2} > 0,\ a \neq 0$.

Theorem (Vinberg)

CJA $J \quad \Leftrightarrow \quad$ self-dual HCC $\mathcal{V} = \mathrm{Int}\{a^2, a \in J\}$.

Let $\mathcal{V} \subset V \simeq V^*$ be a self-dual homogeneous convex cone . Then for any $a \in \mathfrak{g} \subset \mathfrak{gl}(V)$, also $a^t \in \mathfrak{g}$ and then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$$

is a symmetric decomposition , where $\mathfrak{k}$ is the stability subalgebra of a point $o \in \mathcal{V}$.
We identify $\mathfrak{a}$ with V via bijection $\mathfrak{a} \ni a \mapsto T_a o = ao \in V$, T_a is endomorphism, defined by $a \in \mathfrak{g}$.
The multiplication $a \cdot b = T_a b$ defines a structure of compact Jordan algebra in $\mathfrak{a} = V$ with metric
$g(a,b) = g_{cane}(a,b) = \partial^2 \log \phi_e(a,b) = \mathrm{tr}\,(T_{ab})$ and
$\mathcal{V} = Int\{a^2,\ a \in \mathfrak{a}\}$.

List of simple compact Jordan algebras :
algebra of Hermitian matrices :
$Herm_n(\mathbb{K}) = J_n(\mathbb{K})$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and for $n = 3$ over $\mathbb{O}$ and the algebra $\Gamma_{1,n-1} = \mathbb{R}1 \oplus \mathbb{R}^{n-1}$ with multiplication

$$(\alpha, x) \cdot \beta, y = (\alpha\beta + (x, y), \alpha y + \beta x)$$

which corresponds to Lorentz cone in $\mathbb{R}^{1,n-1}$.

Applications:

a) Convex programming

(Nesterov Yu. E., Nemirovskii A.S , Todd J.M. (1994) etc.)

b) Multivariate Statistical Analysis

Wishard, Probability distributions on the space of positively defined matrices (1928).

Anderson, T. W. (1984). An Introduction to Multivariate Statistical Analysis,

S. A. Andersson, G. G. Wojnar, Wishart Distributions on Homogeneous Cones, Journal of Theoretical Probability, Vol. 17, No. 4, 2004 .

c) Principal Geodesic Analysis on Symmetric Spaces:

Statistics of Diffusion Tensors (in MRI)

(P. Thomas Fletcher and Sarang Joshi, Medical Image Display and Analysis Group), 2004.

d) Supergravity. Description of manifolds of scalar multiplets in $d = 3, 4, 5$ $N = 2$ SUGRA, in particular, for study spherically symmetric black holes in supergravity.

Minkowski 4-space is identified with $Herm_2(\mathbb{C}) = J_2(\mathbb{C})$ which is a Jordan algebra .

List of compact simple Jordan algebras

J	$Rot(J)$	$Lor(J)$	$Conf(J)$
$J_n^{\mathbb{R}}$	SO_n	$SL_n(\mathbb{R})$	$Sp_n(\mathbb{R})$
$J_n^{\mathbb{C}}$	SU_n	$SL_n(\mathbb{C})$	$SU_{n,n}$
$J_n^{\mathbb{H}}$	Sp_n	SU^*_{2n}	SO^*_{4n}
$J_3^{\mathbb{O}}$	F_4	E_6^{-26}	E_7^{-25}
$\Gamma_{1,d}$	SO_d	$SO_{d,1}$	$SO_{d,2}$

The selfdual cone $\mathcal{V} = \mathbb{R}^+ \times Lor(J)/Rot(J)$.

$Rot(J) = Aut(J)$ is the group of authomorphisms, $Lor(J)$ is the group, which preserves the norm = determinant, $Conf(I)$ is the structure group, the group , which preserves the norm up to a scaling.

M. Gunaydin, G. Sierra, and P. K. Townsend, The geometry of N = 2 Maxwell-Einstein supergravity and Jordan algebras, Nucl. Phys. B242 (1984) 244.

M. Gunardin, Lectures on Spectrum Generating Symmetries and U-duality in Supergravity , Extremal Black Holes, Quantum Attractors and Harmonic Superspace, JETP. 2009.

A. Marrani Freudenthal Duality in Gravity. From Groups of type E_7 to Prehomogeneous Vector Spaces. in "Group Theory, Probability and Space-Time 2015.

A. Marrani, Non-linear Anti-involutive Symmetry and Black Hole Entropy;
Slides, Mainz Conference "Geometry, Gravity and Sypersymmetry 2017.

III. Theory of homogeneous convex cones $\mathcal{V}$ and domains

Left symmetric algebra associated to a homogeneous convex cone $\mathcal{V}$.

The (absolutely) flat ($T = R = 0$) connection ∂ in a cone $\mathcal{V}$ induces a left invariant flat connection ∇ in the triangular simply transitive group B of $\mathcal{V}$. It corresponds to an endomorphism

$$L : \mathfrak{b} \to \mathrm{End}\,(\mathfrak{b}),\ L_a b = \nabla_a b^*|_e$$

where b^* is the left invariant vector field. Then

$$T(a,b) = L_a b - L_b a - [a,b] = 0,\ R(a,b) = [L_a.L_b] - L_{[a,b]} = 0.$$

The map $a \mapsto \hat{L_a}; x \to L_a x + a$ is an affine representation of the Lie algebra $\mathfrak{b}$ in $\mathfrak{b}$.

Definition (E. Vinberg)

A left-symmetric algebra (LSA)

(also Vinberg algebra, Koszul algebra, pre-Lie algebra, RSA (Matsushima (1968),Gerstenhaber algebra, affine structure in a Lie algebra)

is a non-associative algebra $((\mathfrak{b}), \cdot)$ such that

i) $[a, b] := a \cdot b - b \cdot a = L_a b - L_b a$ is a Lie bracket and

ii) $[L_a, L_b] = L_{[a,b]}$ or $[abc] = [bac]$ where $a \cdot b = L_a b$ and $[abc]$ is the associator.

Theory of LSA and its applications (see survey by D. Burde) LSA appeared in paper by Cayley (1890) as an algebras of rooted trees. This algebra of rooted trees is applied now for renormalisation in QFT and study of Feynman diagrams . It is closely related with rooted tree operads and corresponding Hopf algebra. The algebra of words with 2 letters and algebra of formal vector fields in a vector space are RSA.

LSA $(\mathfrak{g}, a \cdot b = L_a b)$ defines a left invariant affine structures ∇ on a Lie group G with $LieG = \mathfrak{g}, [a, b] = a \cdot b - b \cdot a$.

The affine representation $\hat{L}_a; b \mapsto L_a b + a$ is integrated to an affine action of G in $\mathfrak{g}$ with an open orbit $D = G(0) \subset \mathfrak{g}$.

The connection ∇ is induced by the orbit map $G \to G0$.

LSA is complete iff $D = \mathfrak{g}$.

Y. Benoist constructed first example of (11-dimensional) solvable Lie algebra with no complete LSA structure. It gives a negative answer on a question by J. Milnor. There is no hope to classify LSA, since a symplectic form ω on $\mathfrak{g}$ defines a structure of LSA $L_X Y = -\mathrm{ad}_X^{\omega} Y$

Clans and homogeneous convex cones Let $B \simeq \mathcal{V} \subset V$ be a homogeneous convex cone with simply transitive group B and $a \cdot b = L_a b$ associated LSA structure. The canonical metric $g^{can} = \partial^2 \log \varphi(a)$ induces a left invariant metric g^{can} in $\mathcal{V} = B$. Its value at $T_o\mathcal{V} = T_e(B) = \mathfrak{b} = V$ is given by

$g(a,b) = {g^{can}}_V(a,b) = \partial^2 \ln \varphi_e(a,a) = \mathrm{tr}\, L_{a^2}$

A LSA $(\mathfrak{b}, a \cdot b = L_a b)$ is called a clan if the quadratic form $g(a,a) = \mathrm{tr}\, L_{a \cdot a} > 0,\ a \neq 0$ defines a metric.

Theorem.(Vinberg)

clan (= Compact LSA) $\Leftrightarrow$ homogeneous convex domains

clans with unit $\Leftrightarrow$ homogeneous convex cones.

Matrix algebras and Vinberg T -algebras

Assume that bilinear multiplications

$$\mathfrak{A}_{ij} \times \mathfrak{A}_{jk} \to \mathfrak{A}_{ik},\ i,j = 1,2\cdots n$$

of vector spaces are given. Then the space

$$\mathfrak{A} = \sum_{ij=1}^{n} \mathfrak{A}_{ij} = \{A = ||a_{ij}||,\ a_{ij} \in \mathfrak{A}_{ij}\}$$

of matrices becomes a (matrix) algebra w,r,t, matrix multiplication.
Vinberg T-algebra is a matrix (non associative) algebra with involution
(involutive anti-automorphism $* : \mathfrak{A} \to \mathfrak{A}$ with $*\mathfrak{A}_{ij} = \mathfrak{A}_{ji}$
s.t. subalgebras $\mathfrak{A}_{ii} \simeq \mathbb{R}$ and some axioms holds.

T-algebras and homogeneous convex cones

Axiom implies that the triangular subalgebra

$$\mathfrak{T} = \mathfrak{T}(\mathfrak{A}) = \left\{ B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots b_{1n} \\ 0 & b_{22} & b_{23} & \cdots b_{2n} \\ 0 & 0 & b_{33} & \cdots b_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots b_{nn} \end{pmatrix} \right\}$$

is an associative and the connected component $T(\mathfrak{A})$ of the group of invertible elements of $\mathfrak{T}$ (the triangular group) acts by $B : X \to BXB^*$ on the space $Herm(\mathfrak{A}) = \{X \in \mathfrak{A}, X^* = X\}$ of Hermitian matrices.

Theorem. The orbit $\mathcal{V}(\mathfrak{A}) = T(\mathfrak{A})id = \{BB^*, B \in T(\mathfrak{A})\}$ is a homogeneous convex cone. Any homogeneous convex cone is obtained by this construction from some T-algebra.

Axioms of nilpotent Vinberg N-algebras

Any T-algebra is reconstructed from its (associative nilpotent) upper triangular subalgebra (called N-algebra)

$$\mathfrak{N} = \left\{ \begin{pmatrix} 0 & b_{12} & b_{13} & \cdots b_{1n} \\ 0 & 0 & b_{23} & \cdots b_{2n} \\ 0 & 0 & 0 & \cdots b_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots 0 \end{pmatrix} \right\}$$

which admits a simple axiomatic definition.

N-algebra is an associative nilpotent matrix algebra $\mathfrak{N} = \sum_{i<j} \mathfrak{N}_{ij}$ with Euclidian metric $< .,. >$ such that the bigradation is orthogonal, the multiplication $\mathfrak{N}_{ij} \times \mathfrak{N}_{jk} \to \mathfrak{N}_{ik}$ is an isometric map, i.e.

$$|a_{ij} b_{jk}| = |a_{ij}||b_{jk}|, \; a_{ij} \in \mathfrak{N}_{ij}, \; b_{jk} \in \mathfrak{N}_{jk}$$

and the following condition holds:

If $< a_{ik}, \mathfrak{N} b_{jk} >= 0$, then $< \mathfrak{N} a_{ik}, \mathfrak{N} b_{jk} >= 0$ for $i < j$.

T-algebras of rank 3 An *N*-algebra $\mathfrak{A}_3$ of rank 3 is defined by an isometric map $\mu : U \times V \to W$, where $U = \mathfrak{N}_{12} \simeq \mathbb{R}^p, V = \mathfrak{N}_{23} \simeq \mathbb{R}^q, \ W = \mathfrak{N}_{13} \simeq \mathbb{R}^r$ are Eucledean vector spaces. The dual map is $\mu^* : U \times W \to V$.
The triangular group $T(\mathfrak{A})$ consists of matrices

$$B = \begin{pmatrix} \lambda & a & c \\ 0 & \mu & b \\ 0 & 0 & \nu \end{pmatrix}, \ \lambda, \mu, \nu > 0. a \in U, b \in V. c \in W.$$

It acts in the space of Hermitian matrices, which consists of matrices

$$X = \begin{pmatrix} a & u & w \\ u^* & b & v \\ w^* & v^* & c \end{pmatrix}$$

where $a, b, c \in \mathbb{R}, u \in U, v \in V, w \in W$.

Two particular cases : Self-dual and special algebras .

The T-algebra is called seld-dual iff the cone $\mathcal{V}(\mathfrak{A}_4)$ is self-dual iff $p = q = r$. Then the isometric map is identified with the multiplication in a division algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.
The T-algebra $\mathfrak{A}_3$ is called special iff $q = r$. Then the isometric map

$$\mu + \mu^* : U \times (V \oplus W) \to V + W$$

defines a structure of $Cl(U)$- module in $S = V + W$. They corresponds to homogeneous cones $\mathcal{V} = \mathbb{R}^+ \times \mathcal{V}_1$ The characteristic hypersurfaces $\mathcal{V}_1$ are homogeneous very special real manifolds, Such manifolds are manifolds of scalars (scalar multiplet) in $d = 5$ supergravity with $N = 2$ supersymmetry and via dimensional reductions (r-map and c-map) to homogeneous special Kähler and homogeneous special quaternionic Kähler manifolds. These manifolds describe scalar multiplets in $d = 4$ and $d = 3$ $N = 2$ supergravity.

Dual connection on a Riemannian manifold (M, g)

Definition (W. Blaschke) Let (M, g) be a Riemannian manifold with a connection $\nabla = \nabla^g - C$ with torsion $T = alt(C)$. The conjugated connection ∇^* is defined by the condition

$$Z \cdot g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y))$$

Then $\nabla^*_Z = \nabla^g_Z + C^*_Z$ with torsion $T^{\nabla^*}(X, Y) = C^*_X Y - C^*_Y X$..
Claim. 3-tensor $S := \nabla g$ is a symmetric 3-form iff $T^{\nabla} = T^{\nabla^*}$.

Statistical, Hessian and very special manifolds

Definition (Lauritzen 1987) a) Statistic manifold is a Riemannian manifold (M, g) with torsion free connection $\nabla = \nabla^g - C$ s.t. ∇^* is torsion free $\leftrightarrow C_X = C_X^* \leftrightarrow S = \nabla g$ is symmetric.

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b)(H. Shima) Statistic manifold $(M, g.\nabla)$ with (absolutely) flat connection ∇ is called Hessian manifold (or affine Kähler manifold or dually flat manifold).

c)(B. De Wit A. Van Proeyen) A Hessian manifold $(M, g.\nabla)$ with $\nabla S = 0$ is called (very) special real manifold.

Example A convex domain (in particular, cone) with canonical metric (and also with cone metric) and the flat connection ∂ is a Hessian manifold.

Properties of a statistical manifold $(M, g, \nabla = \nabla^g - C)$.

Definition (N. Chentsov) 1-parameter family of torsion free connection $\nabla^\alpha = \nabla^g - \alpha C$ is called α-family..
Then $\nabla^{-\alpha} = (\nabla^\alpha)^*$. And the curvature is

$$R^\alpha(X, Y) = R^g(X, Y) - \alpha d^{\nabla^g} C(X, Y) + \alpha^2 [C_X, C_Y]$$

$$R^{-\alpha}(X, Y) = R^g(X, Y) + \alpha d^{\nabla^g} C(X, Y) + \alpha^2 [C_X, C_Y] =$$
$$R^\alpha(X, Y) + 2\alpha d^{\nabla^\alpha} C(X, Y) + 4\alpha^2 [C_X, C_Y]$$

Equivalent conditions for (M, g, ∇) to be Hessian..
ii) Tensor C satisfy Codazzi equation .
$(d^{\nabla^g} C)(X, Y) := (\nabla^g_X C)Y - (\nabla^g_Y C)X = 0.$
iii) $d^{\nabla^\alpha} C)(X, Y) = -2\alpha [C_X, C_Y].$
iv) $R^\alpha = R^{-\alpha}$ for some $\alpha \neq 0.$
v) $R^\alpha = R^{-\alpha}$ for any $\alpha.$
vi) Connections ∇, ∇^* - are (strictly) flat . ($T = R = 0$)

Hessian manifolds (M, g, η)

Recall that a Hessian manifolds is Riemanniam manifold (M, g, ∇) with a flat connection ∇ s.t. $S := \nabla g$ is a symmetric 3-form.
Flat Hessian manifolds are closely related with Frobenius manifolds.
In particular, the equation that a Hessian manifold has zero curvature is equivalent to WDVV equation.

Immersion of simply connected Hessian manifold into a domain

Proposition Let (M, g, ∇) be a simply connected Hessian n-manifold Then there exists a canonical affine immersion $\psi : (M, \nabla) \to (\mathbb{R}^n, \partial)$ onto some domain $D = \psi(M) \subset \mathbb{R}^n$, unique up to affine transformations of $\mathbb{R}^n$ s.t. the metric $g = \partial^2 h, \ h \in C^\infty(D)$. M is special real manifold iff h is a cubic polynomial.

The gradients $\mathrm{grad}(x^i)$ of the coordinate functions $x^i = \psi^i$ span a canonical n-dimensional commutative Lie algebra of vector fields. Conversely, a pseudo-Riemannian manifold (M, g) with n pointwise linearly independent gradient vector fields $\mathrm{grad}(x^i)$ is canonically extended to a Hessian manifold (M, g, ∇).

Blaschke immersion of a unimodular simply connected Hessian manifold (M, g, ∇) as an improper affine sphere. A non degenerate convex hypersurface $M \subset \mathbb{A}^{n+1}$ admits a canonical field of normals n such

$$\partial_X Y = \nabla_X Y + g(X, Y)n, \ \partial_X n = SX, \ S \in \mathrm{End}^{sym}(TM)$$

It is called an affine hypersphere if $S = H\mathrm{Id}$ (improper if $H = 0$).
Theorem Let (M, g, ∇) be a simply connected Hessian manifold with $\nabla vol^g = 0$. Then there exists a canonical realisation of (M, g, ∇) as an improper affine hypersphere $\psi : M \to \mathbb{A}^{n+1}$.
If the metric g is complete, it is flat.

Pseudo-Riemannin Hessian geometry associated with a non-degenerate homogeneous function Let $h(x) = h(x^1, \cdots, x^n)$ be a homogeneous positive function of degree d , defined in an open cone $\mathcal{V} \subset V = \mathbb{R}^n$ which is non-degenerate, i.e. $\det Hess(h) \neq 0$. Then

$$g_\lambda = -\frac{1}{d}\partial^2 \log(h)|_{M_\lambda} = -\frac{1}{d}\partial^2 h|_{M_\lambda},\ g_\lambda = \lambda^{1/d} g_1$$

is a metric on $M_\lambda = \{h = \lambda\}$ of a signature (p, q). Then $g^{can} = -\frac{1}{d}\partial^2 \log h$ in $\mathcal{V}$ has signature $(p+1, q) = (+, \cdots, +, -, \cdots, -)$ and the metric $g^{con} = -\frac{1}{d}\partial^2 h-$ has signature $(p, q+1)$. Moreover, $g^{con} = -dr^2 + r^2 g_\lambda$ is the metric of a Lorentz cone over appropriate level set M_λ, If the metric g^{can} is Riemannian, the manifold $(\mathcal{V}, g^{can}, \partial)$ is called Riemannian Hessian manifoild and if, moreover, $h(x)$ is a homogeneous cubic polynomial, it is a special real manifold.

Homogeneous special real manifolds are special rank 3 HCC

A special real manifold $(\mathcal{V}, g^{can}, \partial)$ defined by a homogeneous cubic polynomial is called homogeneous if $\mathcal{V}$ is a homogeneous (convex) cone.

Theorem (De Wit- Van Proeyen, CMP-92; Cortes , TG-96) Any homogeneous special real manifold is a special rank 3 homogeneous convex cone. The cubic polynomial h is described in terms of associated Clifford module.

One can associate with a special real geometry other geometries-

(affine and projective) special Kähler geometry,

special hyper-Kähler and

special quaternionic Kähler geometries.

Special Kähler manifolds

(R.Donagy E.Witten, D. Freed) Affine Special Kähler manifold is a Kähler manifold (N, g, J) with an (absolutely) flat symplectic connection ∇ s.t. ∇J is a symmetric (1,2)-tensor.

Projective spacial Kähler manifold is the projectivisation $\hat{N}/\mathbb{C}^*$ of affine special Kähler cone $\hat{N}$ (with induced Kähler structure).

Special Kähler structure on the base of a complex algebraic integrable system (fibration of a complex symplectic manifold by algebraic Lagrangian tori)

Algebraic completely integrable system is a holomorphic submersion $\pi : X \to M$ from a complex symplectic manifold to a complex manifold M with Lagrangian fibres F and a smooth choice of polarization on F. Then the base manifold M has a structure of affine special Kähler manifold with Gauss-Manin connection.

Special Kähler manifolds as Lagrangian submanifolds. Let $(V = T^*\mathbb{C}^n, \omega)$ be the complex symplectic space with real structure τ, s.t. $V^\tau = T^*\mathbb{R}^n$ and the neutral pseudo-Hermitian metric

$$\gamma(x, y) = \omega(x, \tau y).$$

The graph $\Gamma_{dF} \subset T^*\mathbb{C}^n$, where $F : \mathbb{C}^n \to \mathbb{C}$ is a (generic) holomorphic function with induced by γ Kähler structure, is a special Kähler manifold and locally any SK manifold can be obtained by this construction.

Rigid r-map : from affine special real (M, g, ∇) to affine spacial Kähler (N, g^N, J^N, ∇^N)

Global description (A-Cortes, CMP-08)):

$$N = TM, \;\; T_v N = T_v^h N \oplus T_v^h N = T_{\pi v} M \oplus T_{\pi v} M$$

$$J^N = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}, \qquad g^N = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Rigid c-map : from affine special Kähler $(N, g^N, J^N, \omega^N, \nabla^N)$ to affine spacial hyper-Kähler $(P, g^P, J_1, J_2, J_3 = J_1 J_2)$ (S.Cecotti, S. Ferrara, L. Girardello, Int.J. Mod. Phys-96)

$$P = TN,\ \ T_\xi P = T^h_\xi P \oplus T^h_\xi P \simeq T_{\pi\xi} N \oplus T^*_{\pi\xi} N$$

$$J_1 = \begin{pmatrix} J^N & 0 \\ 0 & J^N \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad g^N = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix}.$$

Conification of special hyper-Kähler manifold and supergravity c-map. A hyper-Kähler manifold (M, g, J_1, J_2, J_3) endowed with a Killing vector field Z, Hamiltonian with respect to the Kähler form $\omega_1 = g \circ J_1$ and rotation two other forms is called a special hyper-Kähler manifold.

The image under c-map of a special Kähler manifold is a special hyper-Kähler manifold.

A hyper-Kähler conification (A-Cortes- Mohaupt, CMP-13) associates with a special hyper-Kähler manifold P a hyper-Kähler cone $\hat{P}$ with action of $\mathbb{H}^*$ by homotheties, s.t. $\hat{P}/\mathbb{H}^*$ is a quaternionic Kähler manifold. The map

$$c_{proj} : N \stackrel{c}{\to} P = TN \stackrel{con}{\to} \hat{P} \to \hat{P}/\mathbb{H}^*$$

is the supergravity c-map (S. Ferrera, S. Sabharwal Nucl. Phys-90), which associates with an affine special Kähler manifold a quaternionic Kähler manifold.

Conification of a special Kähler manifold and supergravity r-map

Recently a conification $con : (N, g, J, \nabla) \to (\hat{N}, \hat{g}, \hat{J}, \hat{\nabla})$ of an affine special Kähler manifold had been defined (V. Cortes, P-S. Dieterich, Th. Mohaupt, Arxiv -17), It associates with an affine special Kähler manifold a special Kähler cone $\hat{N}$ with a homothetic action of $\mathbb{C}^*$, hence , a projective special Kähler manifold $\hat{N}/\mathbb{C}^*$. Composition with the affine r-map gives a projective r-map

$$r_{proj} : M \underset{r}{\to} N \underset{con}{\to} \hat{N} \to \hat{N}/\mathbb{C}^*.$$

from special real manifold to projective special Kähler manifold.