High Order Temperature Model by R.S. Ingarden

High order thermodynamics

- High order moments:
  \[ Q_k = \frac{\partial \Phi(\beta_1, \ldots, \beta_n)}{\partial \beta_k} = \int \frac{-\sum_{k=1}^{n} \langle \beta_k, U^k(\xi) \rangle}{M} \frac{d\omega}{e^{\sum_{k=1}^{n} \langle \beta_k, U^k(\xi) \rangle}} \]

- High order characteristic function:
  \[ \Phi(\beta_1, \ldots, \beta_n) = -\log \int_{M} e^{-\sum_{k=1}^{n} \langle \beta_k, U^k(\xi) \rangle} d\omega \]

- High order temperatures and capacities:
  \[ \beta_k = \frac{\partial S(Q_1, \ldots, Q_n)}{\partial Q_k} \quad K_k = -\frac{\partial Q_k}{\partial \beta_k} \]

- Entropy:
  \[ S(Q_1, \ldots, Q_n) = \sum_{k=1}^{n} \langle \beta_k, Q_k \rangle - \Phi(\beta_1, \ldots, \beta_n) \]

- High order Gibbs density:
  \[ \mathbf{P}_{\text{Gibbs}}(\xi) = e^{\sum_{k=1}^{n} \langle \beta_k, U^k(\xi) \rangle - \Phi(\beta_1, \ldots, \beta_n)} = \int_{M} e^{-\sum_{k=1}^{n} \langle \beta_k, U^k(\xi) \rangle} \frac{d\omega}{e^{-\sum_{k=1}^{n} \langle \beta_k, U^k(\xi) \rangle}} \]
We introduce poly-symplectic extension of Souriau Lie group Thermodynamics based on higher-order model of statistical physics introduced by R.S. Ingarden. This extended model could be used for small data analytics.

Initiated by C. Gunther based on n-symplectic model, it has been shown that the symplectic structure on the phase space remains true, if we replace the symplectic form by a vector valued form, that is called polysymplectic:

- A. Awane, M. Goze, Pfaffian systems, k-symplectic systems. Springer, 2000
Poly-symplectic extension of Souriau Lie Groups Thermodynamics

This extension defines an action of $G$ over $\mathfrak{g}^* \times \ldots \times \mathfrak{g}^*$ called $n$-coadjoint action:

$$Ad^{(n)}_g : G \times (\mathfrak{g}^* \times \ldots \times \mathfrak{g}^*) \rightarrow \mathfrak{g}^* \times \ldots \times \mathfrak{g}^*$$

Let $\mu = (\mu_1, \ldots, \mu_n)$ a poly-momentum, element of $\mathfrak{g}^* \times \ldots \times \mathfrak{g}^*$, we can define a $n$-coadjoint orbit $O_{\mu} = O(\mu_1, \ldots, \mu_n)$ at the point $\mu$, for which the canonical projection:

$$Pr_k : \mathfrak{g}^* \times \ldots \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$$

induces a smooth map between the $n$-coadjoint orbit $O_{\mu}$ and the coadjoint orbit $O_{\mu_k}$:

$$\pi_k : O_{\mu} = O(\mu_1, \ldots, \mu_n) \rightarrow O_{\mu_k}$$

that is a surjective submersion with $\bigcap_{k=1}^{n} Ker T \pi_k = \{0\}$. 
Poly-symplectic extension of Souriau Lie Groups Thermodynamics

- Extending Souriau approach, equivariance of poly-moment could be studied to prove that there is a unique action \( a(\ldots) \) of the Lie group \( G \) on \( \mathfrak{g}^* \times \ldots \times \mathfrak{g}^* \) for which the polymoment map with \( x \in M \) and \( g \in G : J^{(n)} = (J^1, \ldots, J^n) : M \to \mathfrak{g}^* \times \ldots \times \mathfrak{g}^* \) that verifies:

\[
J^{(n)}(\Phi_g(x)) = a(g, J^{(n)}(x)) = Ad_g^{*(n)}(J^{(n)}(x)) + \theta^{(n)}(g)
\]

with \( Ad_g^{*(n)}(J^{(n)}(x)) = (Ad_g^* J^1, \ldots, Ad_g^* J^n) \) and \( \theta^{(n)}(g) = (\theta^1(g), \ldots, \theta^n(g)) \)

- \( \theta^{(n)}(g) \) is a poly-symplectic one-cocycle

- We can also defined poly-symplectic two-cocycle:

\[
\widetilde{\Theta}^{(n)} = (\widetilde{\Theta}^1, \ldots, \widetilde{\Theta}^n) \quad \text{with} \quad \widetilde{\Theta}^k(X, Y) = \langle \Theta^k(X), Y \rangle = J^{(k)}_{[X, Y]} - \{ J^k_X, J^k_Y \}
\]

where \( \Theta^k(X) = T_e \theta^k(X(e)) \)

- the poly-symplectic Souriau-Fisher metric is given by:

\[
g_\beta([\beta, Z_1], Z_2) = \text{diag} \left[ \widetilde{\Theta}^k_{\beta_k}(Z_1, Z_2) \right] , \quad \forall Z_1 \in \mathfrak{g}, \, \forall Z_2 \in \text{Im}(\text{ad}_\beta(\cdot)), \, \beta = (\beta_1, \ldots, \beta_n)
\]

\[
\widetilde{\Theta}^k_{\beta_k}(Z_1, Z_2) = - \frac{\partial \Phi(\beta_1, \ldots, \beta_n)}{\partial \beta^k} = \widetilde{\Theta}^k(Z_1, Z_2) + \langle Q^k, \text{ad}_{Z_1}(Z_2) \rangle
\]
Poly-symplectic extension of Souriau Lie Groups Thermodynamics

- Compared to Souriau model, heat is replaced by previous polysymplectic model:

\[
Q_k = \frac{\partial \Phi(\beta_1, \ldots, \beta_n)}{\partial \beta_k} = \frac{\int U^\otimes k(\xi) e^{-\sum_{k=1}^{n} \langle \beta_k, U^\otimes k(\xi) \rangle} d\omega}{\int e^{-\sum_{k=1}^{n} \langle \beta_k, U^\otimes k(\xi) \rangle} d\omega}
\]

with \( Q = (Q_1, \ldots, Q_n) \in g^* \times \ldots \times g^* \)

- with characteristic function: \( \Phi(\beta_1, \ldots, \beta_n) = -\log \int e^{-\sum_{k=1}^{n} \langle \beta_k, U^\otimes k(\xi) \rangle} d\omega \)

- We extrapolate Souriau results, who proved that \( \int U^\otimes k(\xi) e^{-\langle \beta_k, U^\otimes k(\xi) \rangle} d\omega \) is locally normally convergent using \( \| U^\otimes k \| = \text{Sup}_M \langle E, U \rangle^k \), a multi-linear norm and where \( U^\otimes k = U \otimes U \ldots \otimes U \) is defined as a tensorial product.

- Entropy is defined by Legendre transform of Souriau-Massieu characteristic function:

\[
S(Q_1, \ldots, Q_n) = \sum_{k=1}^{n} \langle \beta_k, Q_k \rangle - \Phi(\beta_1, \ldots, \beta_n) \quad \text{with} \quad \beta_k = \frac{\partial S(Q_1, \ldots, Q_n)}{\partial Q_k}
\]
The Gibbs density could be then extended with respect to high order temperatures by:

\[ p_{Gibbs}(\xi) = e^{-\sum_{k=1}^{n} \langle \beta_k, U^{\otimes k}(\xi) \rangle} \]

\[ \Phi(\beta_1, \ldots, \beta_n) = -\log \int_{M} e^{-\sum_{k=1}^{n} \langle \beta_k, U^{\otimes k}(\xi) \rangle} d\omega \]

with \( U^{\otimes k} = U \otimes U \ldots \otimes U \) and \( \Phi(\beta_1, \ldots, \beta_n) = -\log \int_{M} e^{-\sum_{k=1}^{n} \langle \beta_k, U^{\otimes k}(\xi) \rangle} d\omega \)

where

\[ S(Q_1, \ldots, Q_n) = \sum_{k=1}^{n} \langle \beta_k, Q_k \rangle - \Phi(\beta_1, \ldots, \beta_n) \quad \text{with} \quad \beta_k = \frac{\partial S(Q_1, \ldots, Q_n)}{\partial Q_k} \]

\[ Q_k = \frac{\partial \Phi(\beta_1, \ldots, \beta_n)}{\partial \beta_k} \quad \text{with} \quad Q = (Q_1, \ldots, Q_n) \in \mathfrak{g}^* \times \ldots \times \mathfrak{g}^* \]

\[ \beta_k = \frac{\partial S(Q_1, \ldots, Q_n)}{\partial Q_k} \quad \text{with} \quad \beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{g} \times \ldots \times \mathfrak{g} \]
Comparison of Affine Representation of Lie Group and Lie Algebra in Souriau an Koszul works
Affine representation of Lie group and Lie algebra by Souriau

- Souriau called the Mechanics deduced from his model: “Affine Mechanics”

- Let $G$ be a Lie group and $E$ a finite-dimensional vector space. A map

  $$A : G \rightarrow \text{Aff}(E)$$

  can always be written as:

  $$A(g)(x) = R(g)(x) + \theta(g) \quad \text{with} \quad g \in G, \ x \in E$$

  where the maps $R : G \rightarrow GL(E)$ and $\theta : G \rightarrow E$ are determined by $A$. The map $A$ is an affine representation of $G$ in $E$.

- The map $\theta : G \rightarrow E$ is a one-cocycle of $G$ with values in $E$, for the linear representation $R$; it means that $\theta$ is a smooth map which satisfies, for all $g, h \in G$:

  $$\theta(gh) = R(g)(\theta(h)) + \theta(g)$$
**Affine representation of Lie group and Lie algebra by Souriau**

Let $\mathfrak{g}$ be a Lie algebra and $E$ a finite-dimensional vector space. A linear map $a : \mathfrak{g} \to \text{aff}(E)$ always can be written as:

$$a(X)(x) = r(X)(x) + \Theta(X) \quad \text{with} \quad X \in \mathfrak{g}, x \in E$$

where the linear maps $r : \mathfrak{g} \to \text{gl}(E)$ and $\Theta : \mathfrak{g} \to E$ are determined by $a$. The map $a$ is an affine representation of $G$ in $E$.

The linear map $\Theta : \mathfrak{g} \to E$ is a one-cocycle of $G$ with values in $E$, for the linear representation $r$; it means that $\Theta$ satisfies, for all $X, Y \in \mathfrak{g}$:

$$\Theta([X, Y]) = r(X)(\Theta(Y)) - r(Y)(\Theta(X))$$

$\Theta$ is called the one-cocycle of $\mathfrak{g}$ associated to the affine representation $a$.

The associated cocycle $\Theta : \mathfrak{g} \to E$ is related to the one-cocycle $\theta : G \to E$ by:

$$\Theta(X) = T_e \theta(X(e)) \quad , \quad X \in \mathfrak{g}$$
Equivariance of Souriau Moment Map

There exists a unique affine action $a$ such that the linear part is a coadjoint representation:

$$a : G \times \mathfrak{g}^* \to \mathfrak{g}^*$$

$$a(g, \xi) = \operatorname{Ad}^*_{g^{-1}} \xi + \theta(g)$$

with

$$\left\langle \operatorname{Ad}^*_{g^{-1}} \xi, X \right\rangle = \left\langle \xi, \operatorname{Ad}_{g^{-1}} X \right\rangle$$

that induce equivariance of moment $J$.
Action of Lie Group on a Symplectic Manifold

Let \( \Phi : G \times M \to M \) be an action of Lie Group \( G \) on differentiable manifold \( M \), the fundamental field associated to an element \( X \) of Lie algebra \( \mathfrak{g} \) of group \( G \) is the vectors field \( X_M \) on \( M \):

\[
X_M(x) = \left. \frac{d}{dt} \Phi_{\exp(-tx)}(x) \right|_{t=0}
\]

with \( \Phi_{g_1}(\Phi_{g_2}(x)) = \Phi_{g_1g_2}(x) \) and \( \Phi_e(x) = x \)

\( \Phi \) is Hamiltonian on a symplectic manifold \( M \), if \( \Phi \) is symplectic and if for all \( X \in \mathfrak{g} \), the fundamental field \( X_M \) is globally Hamiltonian.

There is a unique action \( a \) of the Lie group \( G \) on the dual \( \mathfrak{g}^* \) of its Lie algebra for which the moment map \( J \) is equivariant, that means satisfies for each \( x \in M \):

\[
J(\Phi_g(x)) = a(g, J(x)) = Ad_{g^{-1}}^*(J(x)) + \theta(g)
\]

\( \Theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \)

\[
\Theta([X,Y],Z) + \Theta([Y,Z],X) + \Theta([Z,X],Y) = 0
\]
Affine representation of Lie group and Lie algebra by Koszul

Let $\Omega$ be a convex domain in $\mathbb{R}^n$ containing no complete straight lines, and an associated convex cone $V(\Omega) = \{(\lambda x, x) \in \mathbb{R}^n \times \mathbb{R} / x \in \Omega, \lambda \in \mathbb{R}^+\}$

Then there exists an affine embedding:

$$\ell : x \in \Omega \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix} \in V(\Omega)$$

If we consider $\eta$ the group of homomorphism of $A(n, R)$ into $GL(n+1, R)$ given by:

$$s \in A(n, R) \mapsto \begin{bmatrix} f(s) & q(s) \\ 0 & 1 \end{bmatrix} \in GL(n+1, R)$$

and associated affine representation of Lie Algebra:

with $A(n, R)$ the group of all affine transformations of $\mathbb{R}^n$. We have $\eta(G(\Omega)) \subset G(V(\Omega))$ and the pair $(\eta, \ell)$ of the homomorphism $\eta : G(\Omega) \to G(V(\Omega))$ and the map $\ell : \Omega \to V(\Omega)$ is equivariant.
Affine representation of Lie group and Lie algebra by Koszul

Let $G$ a connex Lie Group and $E$ a real or complex vector space of finite dimension, Koszul has introduced an affine representation of $G$ in $E$ such that the following is an affine transformation: $E \rightarrow E$

$a \mapsto sa \quad \forall s \in G$

We set $A(E)$ the set of all affine transformations of a vector space $E$, a Lie Group called affine transformation group of $E$. The set $GL(E)$ of all regular linear transformations of $E$, a subgroup of $A(E)$.

We define a linear representation from $E$ to $GL(E)$:

$f : G \rightarrow GL(E)$

$s \mapsto f(s)a = sa - so \quad \forall a \in E$

and an application from $G$ to $E$:

$q : G \rightarrow E$

$s \mapsto q(s) = so \quad \forall s \in G$

Then we have $\forall s, t \in G$:

$f(s)q(t) + q(s) = q(st)$

$f(s)q(t) + q(s) = s(q(t) - so) + so = s(q(t) = sto = q(st)$
Affine representation of Lie group and Lie algebra by Koszul

- On the contrary, if an application \( q \) from \( G \) to \( E \) and a linear representation \( f \) from \( G \) to \( GL(E) \) verify previous equation, then we can define an affine representation of \( G \) in \( E \), written \((f, q)\):

\[
\text{Aff} (s) : a \mapsto sa = f(s)a + q(s) \quad \forall s \in G, \forall a \in E
\]

- The condition \( f(s)q(t) + q(s) = q(st) \) is equivalent to requiring the following mapping to be an homomorphism: \( \text{Aff} : s \in G \mapsto \text{Aff} (s) \in A(E) \)

- We write \( f \) the linear representation of Lie algebra \( \mathfrak{g} \) of \( G \), defined by \( f \) and \( q \) the restriction to \( \mathfrak{g} \) of the differential to \( q \) (\( f \) and \( q \) the differential of \( f \) and \( q \) respectively), Koszul has proved that:

\[
f(X)q(Y) - f(Y)q(X) = q([X, Y]) \quad \forall X, Y \in \mathfrak{g}
\]

with \( f : \mathfrak{g} \rightarrow gl(E) \) and \( q : \mathfrak{g} \mapsto E \)

Where \( gl(E) \) the set of all linear endomorphisms of \( E \), the Lie algebra of \( GL(E) \).
Conversely, if we assume that \( \mathfrak{g} \) admits an affine representation on \( E \), using an affine coordinate system \( \{x^1, \ldots, x^n\} \) on \( E \), we can express an affine mapping \( v \mapsto f(X)v + q(Y) \) by an \( (n+1) \times (n+1) \) matrix representation:

\[
\text{aff}(X) = \begin{bmatrix}
    f(X) & q(X) \\
    0 & 0
\end{bmatrix}
\]

where \( f(X) \) is a \( n \times n \) matrix and \( q(X) \) is a \( n \) row vector.

If we denote \( \mathfrak{g}_{\text{aff}} = \text{aff}(\mathfrak{g}) \), we write \( G_{\text{aff}} \) the linear Lie subgroup of \( \text{GL}(n+1, R) \) generated by \( \mathfrak{g}_{\text{aff}} \). An element of \( s \in G_{\text{aff}} \) is expressed by:

\[
\text{Aff}(s) = \begin{bmatrix}
    f(s) & q(s) \\
    0 & 1
\end{bmatrix}
\]
Affine representation of Lie Group and Lie Algebra by Souriau and Koszul

<table>
<thead>
<tr>
<th>Souriau Model of Affine Representation of Lie Groups and Algebra</th>
<th>Koszul Model of Affine Representation of Lie Groups and Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(g)(x) = R(g)(x) + \theta(g)$ with $g \in G, x \in E$</td>
<td>$\text{Aff}(s) : a \mapsto sa = f(s)a + q(s)$ for all $s \in G, a \in E$</td>
</tr>
<tr>
<td>$R : G \rightarrow GL(E)$ and $\theta : G \rightarrow E$</td>
<td>$f : G \rightarrow GL(E)$</td>
</tr>
<tr>
<td></td>
<td>$s \mapsto f(s)a = sa - so$ for all $a \in E$</td>
</tr>
<tr>
<td>$\theta(gh) = R(g)(\theta(h)) + \theta(g)$ with $g, h \in G$</td>
<td>$q : G \rightarrow E$</td>
</tr>
<tr>
<td>$\theta : G \rightarrow E$ is a one-cocycle of $G$ with values in $E, \Theta(X) = T_x\theta (X(e))$, $X \in \mathfrak{g}$</td>
<td>$s \mapsto q(s) = so$ for all $s \in G$</td>
</tr>
<tr>
<td>$a(X)(x) = r(X)(x) + \Theta(X)$ with $X \in \mathfrak{g}, x \in E$</td>
<td>$q(st) = f(s)q(t) + q(s)$</td>
</tr>
<tr>
<td>The linear map $\Theta : \mathfrak{g} \rightarrow E$ is a one-cocycle of $G$ with values in $E : \Theta(X) = T_x\theta (X(e))$, $X \in \mathfrak{g}$</td>
<td>$f$ and $q$ the differential of $f$ and $q$ respectively</td>
</tr>
<tr>
<td>$\Theta([X,Y]) = r(X)(\Theta(Y)) - r(Y)(\Theta(X))$</td>
<td>$q ([X,Y]) = f(X)q(Y) - f(Y)q(X)$ for all $X, Y \in \mathfrak{g}$</td>
</tr>
<tr>
<td>none</td>
<td>with $f : \mathfrak{g} \rightarrow gl(E)$ and $q : \mathfrak{g} \mapsto E$</td>
</tr>
<tr>
<td>none</td>
<td>$\text{aff}(X) = \begin{bmatrix} f(X) &amp; q(X) \ 0 &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$\text{Aff}(s) = \begin{bmatrix} f(s) &amp; q(s) \ 0 &amp; 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
Jean-Louis Koszul Lecture in China 1986

- “Introduction à la géométrie symplectique”, in Chinese
- Chuan Yu Ma has written

- This beautiful, modern book should not be absent from any institutional library. ..... During the past eighteen years there has been considerable growth in the research on symplectic geometry. Recent research in this field has been extensive and varied. This work has coincided with developments in the field of analytic mechanics. Many new ideas have also been derived with the help of a great variety of notions from modern algebra, differential geometry, Lie groups, functional analysis, differentiable manifolds and representation theory. [Koszul's book] emphasizes the differential-geometric and topological properties of symplectic manifolds. It gives a modern treatment of the subject that is useful for beginners as well as for experts.
On retrouve dans le livre de Koszul, les équations de Souriau:

17.2. 設 \((M, \omega)\) 是一連通的 Hamilton \(G\)-空間,

\[
\mu : M \rightarrow g^*
\]

是 \((M, \omega)\) 的一個矩射, 則

(i) 任意的 \(s \in G\),

\[
\varphi_\mu(s) = \mu(s) - Ad^*(s)\mu(s)
\]

是 \(g^*\) 中不依賴於點 \(x \in M\) 的一個元素。

(ii) 任意的 \(s, t \in G\) 有

\[
\varphi_\mu(st) = \varphi_\mu(s) + Ad^*(s)\varphi_\mu(t).
\]

(iii) 任意的 \(a, b \in g\) 有

\[
c_\mu(a, b) = \langle d\varphi_\mu(a), b \rangle,
\]

\(c_\mu\) 的定義見 §16.

推論. 从 \(G \times g^*\) 到 \(g^*\) 内的映射

\[
(s, \xi) \mapsto s\xi = Ad^*(s)\xi + \varphi_\mu(s), \quad s \in G, \xi \in g^*,
\]

Développements de Koszul du modèle de Souriau (1/3)

- **Notations:**
  \[ Ad_s a = sas^{-1}, \quad s \in G, a \in \mathfrak{g}, \quad ad_a b = [a, b], \quad a \in \mathfrak{g}, b \in \mathfrak{g} \]
  \[ Ad_s^* = ^tAd_s^{-1}, \quad s \in G \]

- **Propriétés:**
  \[ Ad_{expa} = \exp(-ad_a), \quad a \in \mathfrak{g} \]
  \[ Ad_{expa}^* = \exp^t(ad_a), \quad a \in \mathfrak{g} \]

- **Propriété de l’application moment \( \mu \):**
  \[ \mu : M \rightarrow \mathfrak{g}^* \quad x \mapsto sx, \quad x \in M \]
  \[ \langle d\mu(v), a \rangle = \omega(ax, v) \]
  \[ d\langle Ad_s^* \circ \mu, a \rangle = \langle Ad_s^* d\mu, a \rangle = \langle d\mu, Ad_s^{-1} a \rangle \]
  \[ \langle d\mu(v), Ad_s^{-1} a \rangle = \omega(s^{-1}asx, v) = \omega(asx, sv) = \langle d\mu(sv), a \rangle = (d\langle \mu \circ s_m, a \rangle)(v) \]
  \[ d\langle Ad_s^* \circ \mu, a \rangle = d\langle \mu \circ s_m, a \rangle \Rightarrow d\langle \mu \circ s_m - Ad_s^* \circ \mu, a \rangle = 0 \]
Développements de Koszul du modèle de Souriau (2/3)

- Cocycle symplectique: \( \theta_{\mu}(s) = \mu(sx) - A\text{d}_s^* \mu(x) \), \( s \in G \)

\[
\begin{align*}
\theta_{\mu}(st) &= \mu(stx) - A\text{d}_{st}^* \mu(x) = \theta_{\mu}(s) + A\text{d}_s^* \mu(tx) - A\text{d}_s^* A\text{d}_t^* \mu(x) \\
\theta_{\mu}(st) &= \theta_{\mu}(s) + A\text{d}_s^* \theta_{\mu}(t)
\end{align*}
\]

- Étude de : \( c_{\mu}(a, b) = \langle d\theta_{\mu}(a), b \rangle \), \( a, b \in \mathfrak{g} \)

\[
\begin{align*}
\langle d\mu(ax), b \rangle &= \langle \mu(x), [a, b] \rangle + \langle d\theta_{\mu}(a), b \rangle = \{ \langle \mu, a \rangle, \langle \mu, b \rangle \}(x), \ x \in \mathfrak{M}, a, b \in \mathfrak{g} \\
c_{\mu}(a, b) &= \{ \langle \mu, a \rangle, \langle \mu, b \rangle \} - \langle \mu, [a, b] \rangle = \langle d\theta_{\mu}(a), b \rangle, \ a, b \in \mathfrak{g} \\
c_{\mu}([a, b], c) + c_{\mu}([b, c], a) + c_{\mu}([c, a], b) &= 0, \ a, b, c \in \mathfrak{g} \\
\{ \mu^*(a), \mu^*(b) \} &= \{ \langle \mu, a \rangle, \langle \mu, b \rangle \} = \mu^* \left\{ [a, b] + c_{\mu}(a, b) \right\} = \mu^* \{ a, b \}_\mathfrak{c}_\mu
\end{align*}
\]

- Propriété: \( \mu' = \mu + \varphi \Rightarrow c_{\mu'}(a, b) = c_{\mu}(a, b) - \langle \varphi, [a, b] \rangle \)
Développements de Koszul du modèle de Souriau (3/3)

Action du groupe sur le dual de l’algèbre de Lie:

\[ G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*, (s, \xi) \mapsto s \xi = Ad_s^* \xi + \theta_\mu(s) \]

\[ \mu(sx) = s\mu(x) = Ad_s^* \mu(x) + \theta_\mu(s), \ \forall s \in G, x \in M \]

\[ \theta_\mu(s) = \mu(sx) - Ad_s^* \mu(x) \]

Propriétés:

\[ G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*, (e, \xi) \mapsto e \xi = Ad_e^* \xi + \theta_\mu(e) = \xi + \mu(x) - \mu(x) = \xi \]

\[ (s_1 s_2) \xi = Ad_{s_1 s_2}^* \xi + \theta_\mu(s_1 s_2) = Ad_{s_1}^* Ad_{s_2}^* \xi + \theta_\mu(s_1) + Ad_{s_1}^* \theta_\mu(s_2) \]

\[ (s_1 s_2) \xi = Ad_{s_1}^* (Ad_{s_2}^* \xi + \theta_\mu(s_2)) + \theta_\mu(s_1) = s_1 (s_2 \xi), \ \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^* \]
Euler-Lagrange Equation of Lie Group Thermodynamics
Seminal Paper of Poincaré 1901 on « Euler-Poincaré Equation »


Henri Poincaré proved that when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra

\[ \frac{d}{dt} \frac{dT}{d\eta_s} = \sum c_{ski} \frac{dT}{d\eta_k} \eta_i + \Omega_s. \]
When a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, Henri Poincaré proved that the Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra.

Euler-Poincaré equations can be written under an intrinsic form, without any reference to a particular system of local coordinates, proving that they can be conveniently expressed in terms of the Legendre and momentum maps of the lift to the cotangent bundle of the Lie algebra action on the configuration space.
Euler-Poincaré Equation of Lie Group Thermodynamics

The Lagrangian is a smooth real valued function $L$ defined on the tangent bundle $TM$. To each parameterized continuous, piecewise smooth curve $\gamma : [t_0, t_1] \rightarrow M$, defined on a closed interval $[t_0, t_1]$, with values in $M$, one associates the value at $\gamma$ of the action integral:

$$I(\gamma) = \int_{t_0}^{t_1} L \left( \frac{d\gamma(t)}{dt} \right) dt$$

The partial differential of the function $L : M \times \mathfrak{g} \rightarrow \mathbb{R}$ with respect to its second variable $d_2 \overline{L}$, which plays an important part in the Euler-Poincaré equation, can be expressed in terms of the momentum and Legendre maps:

$$d_2 \overline{L} = p_{\mathfrak{g}}^* \circ \phi' \circ L \circ \phi \quad \text{with} \quad J = p_{\mathfrak{g}}^* \circ \phi' (\implies d_2 \overline{L} = J \circ L \circ \phi)$$

the moment map, $p_{\mathfrak{g}}^* : M \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$

the canonical projection on the 2nd factor, $L : TM \rightarrow T^* M$ the Legendre transform, with $\phi : M \times \mathfrak{g} \rightarrow TM / \phi(x, X) = X_M(x)$

and $\phi^* : T^* M \rightarrow M \times \mathfrak{g}^* / \phi^* (\xi) = (\pi_M (\xi), J(\xi))$
The Euler-Poincaré equation can therefore be written under the form:

\[
\left( \frac{d}{dt} - ad_{V(t)}^* \right) \left( d_2 \bar{L}(\gamma(t), V(t)) \right) = J \circ d_1 \bar{L}(\gamma(t), V(t))
\]

\[
\left( \frac{d}{dt} - ad_{V(t)}^* \right) \left( J \circ L \circ \phi(\gamma(t), V(t)) \right) = J \circ d_1 \bar{L}(\gamma(t), V(t)) \quad \text{with} \quad \frac{d\gamma(t)}{dt} = \phi(\gamma(t), V(t))
\]

\[
H(\xi) = \left\langle \xi, L^{-1}(\xi) \right\rangle - L \left( L^{-1}(\xi) \right), \quad \xi \in T^*M, \quad L : TM \to T^*M, \quad H : T^*M \to R
\]

Following the remark made by Poincaré at the end of his note, the most interesting case is when the map \( \bar{L} : M \times g \to R \) only depends on its second variable \( X \in g \). The Euler-Poincaré equation becomes:

\[
\left( \frac{d}{dt} - ad_{V(t)}^* \right) \left( d\bar{L}(V(t)) \right) = 0 \quad \Rightarrow \quad d\bar{L}(V(t)) = d_2 \bar{L}(\gamma(t), V(t))
\]
**Euler-Poincaré Equation of Lie Group Thermodynamics**

- We can use analogy of structure when the convex Gibbs ensemble is homogeneous. We can then apply Euler-Poincaré equation for Lie Group Thermodynamics. Considering Clairaut equation:

\[ s(Q) = \langle \beta, Q \rangle - \Phi(\beta) = \langle \Theta^{-1}(Q), Q \rangle - \Phi(\Theta^{-1}(Q)) \]

with \( Q = \Theta(\beta) = \frac{\partial \Phi}{\partial \beta} \in \mathfrak{g}^* \), \( \beta = \Theta^{-1}(Q) \in \mathfrak{g} \)

- A Souriau-Euler-Poincaré equation can be elaborated for Souriau Lie Group Thermodynamics:

\[
\frac{dQ}{dt} = a_d^* Q \quad \text{and} \quad \frac{d}{dt} (Ad_g^* Q) = 0
\]

- An associated equation on Entropy is:

\[
\frac{ds}{dt} = \left\langle \frac{d\beta}{dt}, Q \right\rangle + \langle \beta, a_d^* Q \rangle - \frac{d\Phi}{dt}
\]

That reduces to:

\[
\frac{ds}{dt} = \left\langle \frac{d\beta}{dt}, Q \right\rangle - \frac{d\Phi}{dt}
\]

Due to

\[
\langle \xi, ad^*_V X \rangle = -\langle ad_v^* \xi, X \rangle \Rightarrow \langle \beta, a_d^* Q \rangle = \langle Q, a_d^* \beta \rangle = 0
\]
Poincaré-Cartan Integral Invariant of Lie Group Thermodynamics

Analogies between Geometric Mechanics & Geometric Lie Group Thermodynamics, provides the following similarities of structures:

\[
\begin{align*}
\dot{q} &\leftrightarrow \beta \\
p &\leftrightarrow Q \\
L(q) &\leftrightarrow \Phi(\beta) \\
H(p) &\leftrightarrow s(Q) \\
H = p\dot{q} - L &\leftrightarrow s = \langle Q, \beta \rangle - \Phi
\end{align*}
\]

We can then consider a similar Poincaré-Cartan-Souriau Pfaffian form:

\[
\omega = p.dq - H.dt \leftrightarrow \omega = \langle Q, (\beta.dt) \rangle - s.dt = (\langle Q, \beta \rangle - s).dt = \Phi(\beta).dt
\]

This analogy provides an associated Poincaré-Cartan Integral Invariant:

\[
\int_{C_a} p.dq - H.dt = \int_{C_b} p.dq - H.dt \quad \text{transforms in} \quad \int_{C_a} \Phi(\beta).dt = \int_{C_b} \Phi(\beta).dt
\]

For Thermodynamics, we can then deduce an Euler-Poincaré Variational Principle: The Variational Principle holds on \(q\), for variations \(\delta \beta = \dot{\eta} + [\beta, \eta]\), where \(\eta(t)\) is an arbitrary path that vanishes at the endpoints, \(\eta(a) = \eta(b) = 0\):

\[
\delta \int_{t_0}^{t_1} \Phi(\beta(t)).dt = 0
\]
Compatibility with Gauge Model of Balian-Valentin
Entropy $S$ is an extensive variable $q^0 = S(q^1,...,q^n)$ depending on $n$ independent extensive/conservative quantities characterizing the system.

The $n$ intensive variables $\gamma_i$ are defined as the partial derivatives:

$$\gamma_i = \frac{\partial S(q^1,...,q^n)}{\partial q^i}$$

Balian has introduced a non-vanishing gauge variable which multiplies all the intensive variables, defining a new set of variables: $p_i = -p_0 \gamma_i$, $i = 1,...,n$.

The 2n+1-dimensional space is thereby extended into a 2n+2-dimensional thermodynamic space $T$ spanned by the variables $p_i, q^i$ with $i = 0,1,...,n$, where the physical system is associated with a n+1-dimensional manifold $M$ in $T$, parameterized for instance by the coordinates $q^1,...,q^n$ and $p_0$. 


Compatible Balian Gauge Theory of Thermodynamics

- the contact structure in 2n+1 dimension: \( \tilde{\omega} = dq^0 - \sum_{i=1}^{n} \gamma_i dq^i \)

- is embedded into a symplectic structure in 2n+2 dimension, with 1-form, as symplectization: \( \omega = \sum_{i=0}^{n} p_i dq^i \)

- The n+1-dimensional thermodynamic manifolds \( M \) are characterized by: \( \omega = 0 \). The 1-form induces then a symplectic structure on \( T \): \( d\omega = \sum_{i=0}^{n} dp_i \wedge dq^i \)

- The concavity of the entropy \( S(q^1, \ldots, q^n) \) as function of the extensive variables, expresses the stability of equilibrium states. It entails the existence of a metric structure in the n-dimensional space \( q_i \):
  \[ ds^2 = -d^2 S = -\sum_{i,j=1}^{n} \frac{\partial^2 S}{\partial q^i \partial q^j} dq^i dq^j \]
  which defines a distance between two neighboring thermodynamic states:
  \[ d\gamma_i = \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q^i \partial q^j} dq^j \]
  \[ ds^2 = -\sum_{i=1}^{n} d\gamma_i dq_i = \frac{1}{p_0} \sum_{i=0}^{n} dp_i dq^i \]

\[ \]
We can observe that this Gauge Theory of Thermodynamics is compatible with Souriau Lie Group Thermodynamics, where we have to consider the Souriau vector:

\[
\beta = \begin{bmatrix}
\gamma_1 \\
\vdots \\
\gamma_n
\end{bmatrix}
\]

transformed in a new vector

\[
p_i = -p_0 \cdot \gamma_i
\]

\[
p = \begin{bmatrix}
-p_0 \gamma_1 \\
\vdots \\
-p_0 \gamma_n
\end{bmatrix} = -p_0 \cdot \beta
\]
Links with Natural Exponential Families Invariant by a Group: Casilis and Letac
NEF (Natural Exponential Families): Letac & Casalis

Let $E$ a vector space of finite size, $E^*$ its dual. $\langle \theta, x \rangle$ Duality braket with $(\theta, x) \in E^* \times E$. $\mu$ Positive Radon measure on $E$, Laplace transform is:

$L_\mu : E^* \rightarrow [0, \infty]$ with $\theta \mapsto L_\mu(\theta) = \int e^{\langle \theta, x \rangle} \mu(dx)$

Transformation $k_\mu(\theta)$ defined on $\Theta(u)^E$ interior of $D_\mu = \{ \theta \in E^*, L_\mu < \infty \}$

$k_\mu(\theta) = \log L_\mu(\theta)$

Natural exponential families are given by:

$F(\mu) = \{ P(\theta, \mu)(dx) = e^{\langle \theta, x \rangle - k_\mu(\theta)} \mu(dx), \theta \in \Theta(\mu) \}$

Injective function (domain of means):

$k'_\mu(\theta) = \int xP(\theta, \mu)\mu(dx)$

And the inverse function:

$\psi_\mu : M_F \rightarrow \Theta(\mu)$ with $M_F = \text{Im}(k'_\mu(\Theta(\mu)))$

Covariance operator:

$V_F(m) = k''_{\mu}(\psi_\mu(m)) = (\psi'_\mu(m))^{-1}$, $m \in M_F$
NEF (Natural Exponential Families): Letac & Casalis

Measure generated by a family $F$:

$$F(\mu) = F(\mu') \iff \exists (a, b) \in E^* \times \mathbb{R}, \text{such that } \mu'(dx) = e^{\{a, x\} + b} \mu(dx)$$

Let $F$ an exponential family of $E$ generated by $\mu$ and $\varphi : x \mapsto g_\varphi x + v_\varphi$ with $g_\varphi \in GL(E)$ automorphisms of $E$ and $v_\varphi \in E$, then the family

$$\varphi(F) = \{\varphi(P(\theta, \mu)), \theta \in \Theta(\mu)\}$$

is an exponential family of $E$

generated by $\varphi(\mu)$

Definition: An exponential family $F$ is invariant by a group $G$ (affine group of $E$), if $\forall \varphi \in G, \varphi(F) = F$:

$$\forall \mu, F(\varphi(\mu)) = F(\mu)$$

(the contrary could be false)
Theorem (Casalis): Let $F = F(\mu)$ an exponential family of $E$ and $G$ affine group of $E$, then $F$ is invariant by $G$ if and only:

$\exists a : G \to E^*, \exists b : G \to R$, such that:

$$a(\varphi \varphi') = g^{-1}_\varphi a(\varphi') + a(\varphi)$$

$$b(\varphi \varphi') = b(\varphi) + b(\varphi') - \langle a(\varphi'), g^{-1}_\varphi v_\varphi \rangle$$

$$\forall \varphi \in G, \varphi(\mu)(dx) = e^{\langle a(\varphi), x \rangle + b(\varphi)} \mu(dx)$$

When $G$ is a linear subgroup, $b$ is a character of $G$, $a$ could be obtained by the help of Cohomology of Lie groups.
NEF (Natural Exponential Families): Letac & Casalis

If we define action of $G$ on $E^*$ by:  
\[ g.x = g^{-1}x, \quad g \in G, \quad x \in E^* \]

we can verify that:  
\[ a(g_1 g_2) = g_1(a(g_2)) + a(g_1) \]

the action $a$ is an inhomogeneous 1-cocycle:  
\[ \forall n > 0, \text{ let the set of all functions from } G^n \text{ to } E^*, \ \mathcal{S}(G^n, E^*) \text{ called inhomogenous n-cochains,} \]

then we can define the operators:  
\[ d^n : \mathcal{S}(G^n, E^*) \to \mathcal{S}(G^{n+1}, E^*) \]

\[ d^n F(g_1, \ldots, g_{n+1}) = g_1 F(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i F(g_1, g_2, \ldots, g_i g_{i+1}, \ldots, g_n) \]

\[ + (-1)^{n+1} F(g_1, g_2, \ldots, g_n) \]
NEF (Natural Exponential Families): Letac & Casalis

Let \( Z^n(G, E^*) = \ker(d^n), B(G, E^*) = \text{im}(d^{n-1}) \), with \( Z^n \) inhomogeneous \( n \)-cocycles, the quotient \( H^n(G, E^*) = Z^n(G, E^*) / B^n(G, E^*) \) is the Cohomology Group of \( G \) with value in \( E^* \). We have:

\[
\begin{align*}
  d^0 : E^* &\to \mathcal{Z}(G, E^*) \\
  x &\mapsto (g \mapsto g.x - x) \\
  d^1 : \mathcal{Z}(G, E^*) &\to \mathcal{Z}(G^2, E^*) \\
  F &\mapsto d^1 F \quad d^1 F(g_1, g_2) = g_1.F(g_2) - F(g_1g_2) + F(g_1) \\
  Z^1 &= \{ F \in \mathcal{Z}(G, E^*); F(g_1g_2) = g_1.F(g_2) + F(g_1), \forall (g_1, g_2) \in G^2 \} \\
  B^1 &= \{ F \in \mathcal{Z}(G, E^*); \exists x \in E^*, F(g) = g.x - x \}
\end{align*}
\]
NEF (Natural Exponential Families): Letac & Casalis

When the Cohomology Group $H^1(G, E^*) = 0$ then $Z^1(G, E^*) = B^1(G, E^*)$

$\Rightarrow \exists c \in E^*$, such that $\forall g \in G, a(g) = \left(I_d - t g^{-1}\right)c$

Then if $F = F(\mu)$ is an exponential family invariant by $G$, $\mu$ verifies

$\forall g \in G, g(\mu)(d\mu) = e^{\langle c, x \rangle - \langle c, g^{-1} x \rangle + b(g)} \mu(d\mu)$

$\forall g \in G, g \left(e^{\langle c, x \rangle} \mu(\mu)\right) = e^{b(g)} e^{\langle c, x \rangle} \mu(\mu)$ with $\mu_0(\mu) = e^{\langle c, x \rangle} \mu(\mu)$

For all compact Group, $H^1(G, E^*) = 0$ and we can express $a$

$A: G \rightarrow GA(E)$

$\forall (g, g') \in G^2, A_{gg'} = A_g A_{g'}$

$g \mapsto A_g$, $A_g(\theta) = t g^{-1} \theta + a(g)$ A(G) compact sub-group of GA(E)

$\exists$ fixed point $\Rightarrow \forall g \in G, A_g(c) = t g^{-1} c + a(g) = c \Rightarrow a(g) = \left(I_d - t g^{-1}\right)c$
LIE GROUP
For Exponential Families in Information Geometry
Lie Group, Lie Algebra, Dual Lie Algebra

Arnold, Vladimir, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Annales de l'institut Fourier, 16 no. 1 (1966), p. 319-361:
http://archive.numdam.org/article/AIF_1966__16_1_319_0.pdf

Vladimir Arnold Joke:
« Dans ce qui suit, j'ai tâché conformément à l'appel de N. Bourbaki, de substituer toujours les calculs aveugles aux idées lucides d'Euler »

From Vladimir Arnold Paper
Affine Group Action

Consider the General Linear Group \( GL(n) \) consisting of the invertible \( nxn \) matrices, that is a topological group acting linearly on \( R^n \) by:

\[
GL(n) \times R^n \rightarrow R^n
\]

\[
(A, x) \mapsto Ax
\]

The Group \( GL(n) \) is a Lie group, is a subgroup of the General affine group \( GA(n) \), composed of all pairs \( (A, \nu) \) where \( A \in GL(n) \) and \( \nu \in R^n \), the group operation given by:

\[
(A_1, \nu_1)(A_2, \nu_2) = (A_1A_2, A_1\nu_2 + \nu_1)
\]

Restricting \( A \) to have positive determinant one obtains the Positive General affine group \( GA_+ (n) \) that acts transitively on \( R^n \) by:

\[
((A, \nu), x) \mapsto Ax + \nu
\]

Given a positive semidefinite matrix \( R \), according to the spectral theorem, the continuous functional calculus can be applied to obtain a matrix \( R^{1/2} \) such that \( R^{1/2} \) is itself positive and \( R^{1/2} R^{1/2} = R \). The operator \( R^{1/2} \) is the unique non-negative square root of \( R \).
**Affine Group Action**

\[ N_n = \left\{ \mathcal{N}(\mu, \Sigma) | \mu \in \mathbb{R}^n, \Sigma \in \text{Sym}^+ \right\} \] the class of regular multivariate normal distributions, where $\mu$ is the mean vector and $\Sigma$ is the (symmetric positive definite) covariance matrix, is invariant under the transitive action of $GA(n)$. The induced action of $GA(n)$ on $\mathbb{R}^n \times \text{Sym}^+ \mathbb{R}^n$ is then given by:

\[ GA(n) \times \left( \mathbb{R}^n \times \text{Sym}^+ \mathbb{R}^n \right) \to \mathbb{R}^n \times \text{Sym}^+ \mathbb{R}^n \]

\[ ((A, \nu), (\mu, \Sigma)) \mapsto (A\mu + \nu, A\Sigma A^T) \]

and:

\[ GA(n) \times \mathbb{R}^n \to \mathbb{R}^n \]

\[ ((A, \nu), x) \mapsto Ax + \nu \]

As the isotropy group of $(0, I_n)$ is equal to $O(n)$, we can observe that:

\[ N_n = GA(n) / O(n) \]

\[ N_n \] is an open subset of the vector space $T_n = \left\{ (\eta, \Omega) | \eta \in \mathbb{R}^n, \Omega \in \text{Sym}_n \right\}$ and is a differentiable manifold, where the tangent space at any point may be identified with $T_n$. 

This document may not be reproduced, modified, adapted, published, translated, in any way, in whole or in part or disclosed to a third party without the prior written consent of Thales
Affine group for Multivariate Gaussian Law

$X \approx \mathcal{N}(0, I)$

$Y \approx \mathcal{N}(m, R)$

Action of Affine Lie Group

$[Y] = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} [X] = \begin{bmatrix} R^{1/2}X + m \\ 1 \end{bmatrix}$,

$\left\{ (m, R) \in \text{Sym}(n) \times \mathbb{R}^n \right\}$

$M = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \in G_{aff}$

$X \approx \mathcal{N}(0, I) \rightarrow \mathcal{N}(m, R)$

$\mathbb{R}^{1/2}$: Cholesky Decomposition of $R$

$\mathbb{R}^{1/2}$ is an element of the group of triangular matrix with positive elements on the diagonal

Lie Group Everywhere: example of multivariate gaussian law
Affine Group (Lie Group) and associated Lie Algebra

**Affine Group in case of Multivariate Gaussian case**

\[
\begin{bmatrix}
Y \\
1
\end{bmatrix} = \begin{bmatrix}
R^{1/2} & m \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
X \\
1
\end{bmatrix} = \begin{bmatrix}
R^{1/2}X + m \\
1
\end{bmatrix}
\]

\[
(m, R) \in R^n \times \text{Sym}(n)
\]

\[
M = \begin{bmatrix}
R^{1/2} & m \\
0 & 1
\end{bmatrix} \in G_{aff}
\]

\[X \approx \mathcal{N}(0, I) \rightarrow Y \approx \mathcal{N}(m, R)\]

**Lie Group properties**

\[
M_1.M_2 = \begin{bmatrix}
R_1^{1/2} & m_1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
R_2^{1/2} & m_2 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
R_1^{1/2}R_2^{1/2} & R_1^{1/2}m_2 + m_1 \\
0 & 1
\end{bmatrix}
\]

\[
M_2.M_1 = \begin{bmatrix}
R_2^{1/2} & m_2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
R_1^{1/2} & m_1 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
R_2^{1/2}R_1^{1/2} & R_2^{1/2}m_1 + m_2 \\
0 & 1
\end{bmatrix}
\]

\[
\begin{align*}
M_1.M_2 & \in G_{aff} \\
M_2.M_1 & \in G_{aff} \\
M_1.M_2 & \neq M_2.M_1 \\
M_1.(M_2.M_3) & = (M_1.M_2).M_3 \\
M_1.I & = M_1
\end{align*}
\]
**Affine Group (Lie Group) and associated Lie Algebra**

- **Inverse element:**
  
  \[
  M = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \Rightarrow M_R^{-1} = M_L^{-1} = M^{-1} = \begin{bmatrix} R^{-1/2} & -R^{-1/2}m \\ 0 & 1 \end{bmatrix} \in G_{aff}
  \]

- **Lie Algebra:** \( g \)

  \[
  L_G : \begin{cases} 
  G_{aff} \to G_{aff} \\
  M \mapsto L_M N = M.N 
  \end{cases} \quad \text{and} \quad R_G : \begin{cases} 
  G_{aff} \to G_{aff} \\
  M \mapsto R_M N = N.M 
  \end{cases}
  \]

  \[
  \dot{\gamma}(t) = \begin{bmatrix} R^{1/2} (t) & m(t) \\ 0 & 1 \end{bmatrix}, \quad \dot{\gamma}(t) = \begin{bmatrix} \dot{R}^{1/2} (t) & \dot{m}(t) \\ 0 & 0 \end{bmatrix} \Rightarrow \Gamma_L(t) = L_M^{-1} (\gamma(t)) = \begin{bmatrix} R^{-1/2} \dot{R}^{1/2} (t) & R^{-1/2} (m(t) - m) \\ 0 & 1 \end{bmatrix}
  \]

  \[
  \dot{\Gamma}_L(t) \bigg|_{t=0} = \begin{bmatrix} R^{-1/2} \dot{R}^{1/2} (0) & R^{-1/2} \dot{m}(0) \\ 0 & 1 \end{bmatrix} = dL_M^{-1} \dot{\gamma}(0) = dL_M^{-1} \dot{M}
  \]

  \[
  dL_M^{-1} : T_M (G) \to g_L
  \]

  \[
  \dot{M} \mapsto \Omega_L = dL_M^{-1} \dot{M} = M^{-1} \dot{M} = \begin{bmatrix} R^{-1/2} \dot{R}^{1/2} & R^{-1/2} \dot{m} \\ 0 & 0 \end{bmatrix}
  \]
Affine Group (Lie Group) and associated Lie Algebra

**Lie Algebra on the right and on the left**

\[ dL_{M^{-1}} : T_M(G) \rightarrow g_L \]

\[ \dot{M} \mapsto \Omega_L = dL_{M^{-1}} \dot{M} = M^{-1} \dot{M} = \begin{bmatrix} R^{-1/2} \dot{R}^{1/2} & R^{-1/2} \dot{m} \\ 0 & 0 \end{bmatrix} \]

\[ dR_{M^{-1}} : T_M(G) \rightarrow g_R \]

\[ \dot{M} \mapsto \Omega_R = dR_{M^{-1}} \dot{M} = \dot{M}M^{-1} = \begin{bmatrix} R^{-1/2} \dot{R}^{1/2} & \dot{m} - R^{-1/2} \dot{R}^{1/2} \dot{m} \\ 0 & 0 \end{bmatrix} \]

\[
\begin{bmatrix}
X(t) \\
1
\end{bmatrix} = M \begin{bmatrix}
x \\
1
\end{bmatrix} \Rightarrow \begin{bmatrix}
\dot{X}(t) \\
0
\end{bmatrix} = \Omega_R \begin{bmatrix}
X(t) \\
1
\end{bmatrix} \quad \text{with } x \text{ fixed}
\]

\[
\begin{bmatrix}
x(t) \\
1
\end{bmatrix} = M^{-1} \begin{bmatrix}
X \\
1
\end{bmatrix} \Rightarrow \begin{bmatrix}
\dot{x}(t) \\
0
\end{bmatrix} = -\Omega_L \begin{bmatrix}
X \\
1
\end{bmatrix} \quad \text{with } X \text{ fixed}
\]
Affine Group (Lie Group) and associated Lie Algebra

- **Conjugation Action**

  \[ AD : G \times G \rightarrow G \]
  \[ M, N \mapsto AD_M N = M \cdot N \cdot M^{-1} \]

- **Adjoint Operator on Lie Group**

  \[ Ad : G \times g \rightarrow g \]

  \[ M, n \mapsto Ad_M n = M \cdot n \cdot M^{-1} = \frac{d}{dt} \bigg|_{t=0} (AD_M N(t)) \quad \text{with} \quad \begin{cases} N(0) = I \\ \dot{N}(0) = n \in g \end{cases} \]

\[
\begin{align*}
M_1 &= \begin{bmatrix} R_1^{1/2} & m_1 \\ 0 & 1 \end{bmatrix}, & M_2 &= \begin{bmatrix} R_2^{1/2} & m_2 \\ 0 & 1 \end{bmatrix} \\
AD_{M_1} M_2 &= \begin{bmatrix} R_2^{1/2} & -R_2^{1/2} m_1 + R_1^{1/2} m_2 + m_1 \\ 0 & 1 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
n_{2L} &= \begin{bmatrix} -R_2^{1/2} \dot{R}_2^{1/2} & R_2^{-1/2} m_2 \\ 0 & 0 \end{bmatrix}, \quad n_{2R} &= \begin{bmatrix} -R_2^{1/2} \dot{R}_2^{1/2} & R_2^{1/2} m_2 + \dot{R}_2^{1/2} m_2 + R_2^{1/2} m_2 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
Ad_{M_1} n_{2L} &= n_{2R} \quad \text{and} \quad Ad_{M_1} n_{2R} &= \begin{bmatrix} -R_2^{1/2} \dot{R}_2^{1/2} & R_2^{1/2} m_2 + \dot{R}_2^{1/2} m_2 + R_2^{1/2} m_2 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]
Affine Group (Lie Group) and associated Lie Algebra

### Adjoint operator on Lie Algebra

\[ \text{ad} : g \times g \to g \]

\[ n, m \mapsto \text{ad}_m n = m.n - n.m = \left. \frac{d}{dt} \right|_{t=0} (\text{Ad}_M n(t)) = [m, n] \]

with \[
\begin{align*}
\dot{N}(0) &= n \in g \\
\dot{M}(0) &= m \in g 
\end{align*}
\]

\[ n_{1L} = \begin{bmatrix} R_1^{-1/2} \dot{R}_1^{1/2} & R_1^{-1/2} \dot{m}_1 \\ 0 & 0 \end{bmatrix}, \quad n_{2L} = \begin{bmatrix} R_2^{-1/2} \dot{R}_2^{1/2} & R_2^{-1/2} \dot{m}_2 \\ 0 & 0 \end{bmatrix} \]

\[ \text{ad}_{n_{1L}} n_{2L} = [n_{1L}, n_{2L}] = \begin{bmatrix} 0 & R_1^{-1/2} \left( \dot{R}_1^{1/2} m_2 - \dot{R}_2^{1/2} \dot{m}_1 \right) R_2^{-1/2} \\ 0 & 0 \end{bmatrix} \]

\[ \text{ad}_{n_{1R}} n_{2R} = [n_{1R}, n_{2R}] = \begin{bmatrix} 0 & R_1^{-1/2} \dot{R}_1^{1/2} \left( -R_2^{-1/2} \dot{R}_2^{1/2} m_2 + \dot{m}_2 \right) - R_2^{-1/2} \dot{R}_2^{1/2} \left( -R_1^{-1/2} \dot{R}_1^{1/2} m_1 + \dot{m}_1 \right) \\ 0 & 0 \end{bmatrix} \]
Affine Group (Lie Group) and associated Lie Algebra

Moments Maps

\[ n_L = \begin{bmatrix} R^{-1/2} \hat{R}^{1/2} & R^{-1/2} \hat{m} \\ 0 & 0 \end{bmatrix} \]

\[ \Pi_L = \frac{\partial E_L}{\partial n_L} = n_L \]

\[ \langle .. \rangle : g^* \times g \rightarrow R \\
\langle k, n \rangle = Tr(k^T n) \]

\[ E_L = \frac{1}{2} \langle n_L, n_L \rangle = \frac{1}{2} Tr[n_L^T n_L] = \frac{1}{2} (Tr(R^{-1} \hat{R}) + \hat{m}^T R^{-1} \hat{m}) \]

\[ E_L = \langle \Pi_L, n_L \rangle = \langle \Pi_L, M^{-1} n_R M \rangle = \langle \Pi_R, n_R \rangle \]

\[ \Pi_L : g \rightarrow g^* \]

\[ n_L \mapsto \Pi_L \]