Rolling Symmetric Spaces

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Main goals

- Present a unifying theory to describe rolling motions of Riemannian symmetric homogeneous spaces
- Establish the connection between the kinematic equations of rolling and the Lie algebra decomposition associated to the symmetric space

Outline

1. **Definition of Rolling**
2. **Homogenous Symmetric Spaces**
3. **Rolling Riemannian Symmetric Homogeneous Spaces**
4. **Illustrative examples**
   - Euclidean sphere
   - Lorentzian sphere
   - Grassmann manifold
5. **Conjecture**

Rolling map

$\widetilde{M}$ Riemannian $m-$manifold $\quad (\widetilde{M} = \mathbb{R}^m)$

$\widetilde{G}$ group of isometries on $\widetilde{M}$ $\quad (\widetilde{G} = SO(m) \ltimes \mathbb{R}^m)$

M and M_0 $n-$manifolds isometrically embedded in $\widetilde{M}$

Definition

[Sharpe (1997)]

$\chi : I \subset \mathbb{R} \longrightarrow \widetilde{G}$ is a **rolling map** of M on M_0 along a smooth curve $\sigma : I \longrightarrow M$ (rolling curve) if, for all $t \in I$,

rolling:

- $\chi(t) \cdot \sigma(t) := \sigma_0(t) \in M_0$ (development curve of σ)
- $T_{\chi(t)\cdot\sigma(t)}\big(\chi(t)(M)\big) = T_{\chi(t)\cdot\sigma(t)} M_0$.

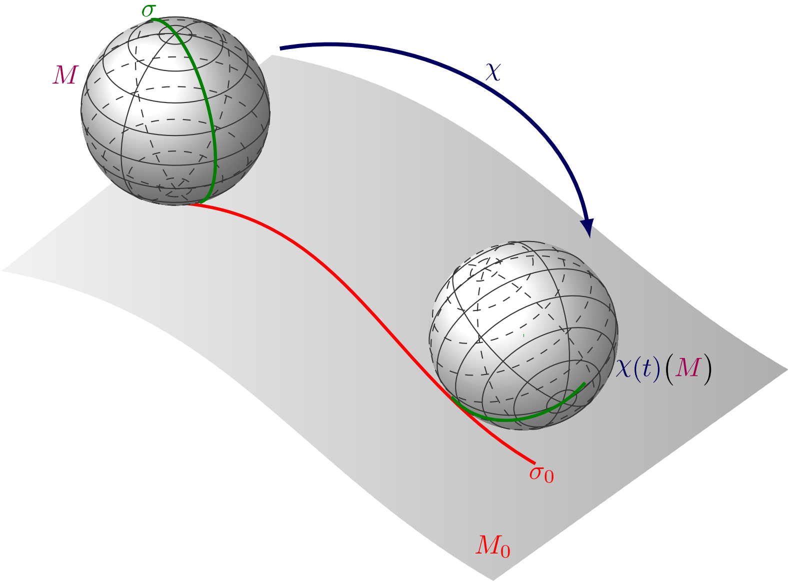
no-slip:

- $\dot{\sigma}_0(t) = d_{\sigma(t)}\chi(t) \cdot \dot{\sigma}(t)$

no-twist:

- $d_{\sigma_0(t)}\big(\dot{\chi}(t)\chi(t)^{-1}\big)\big(T_{\sigma_0(t)}M_0\big) \subset T^{\perp}_{\sigma_0(t)}M_0$
- $d_{\sigma_0(t)}\big(\dot{\chi}(t)\chi(t)^{-1}\big)\big(T^{\perp}_{\sigma_0(t)}M_0\big) \subset T_{\sigma_0(t)}M_0$.

Illustration of rolling



Structure of $d(\dot{\chi}\chi^{-1})$

For an appropriate orthonormal basis of $T_{\sigma_0(t)}\widetilde{M}$

$\left[T_{\sigma_0(t)}\widetilde{M} = T_{\sigma_0(t)}M_0 \oplus T^{\perp}_{\sigma_0(t)}M_0\right]$

$$d_{\sigma_0(t)}\left(\dot{\chi}(t)\chi(t)^{-1}\right) = \begin{array}{c} \begin{array}{cc} T_{\sigma_0(t)}M_0 & T^{\perp}_{\sigma_0(t)}M_0 \end{array} \\ \left[\begin{array}{c|c} 0 & X_{n\times(m-n)} \\ \hline -X^T_{(m-n)\times n} & 0 \end{array}\right] \end{array} \begin{array}{l} T_{\sigma_0(t)}M_0 \\ T^{\perp}_{\sigma_0(t)}M_0 \end{array}$$

We will show that on a **symmetric space** M, the structure of $d(\dot{\chi}\chi^{-1})$ is captured from the **decomposition of the Lie algebra** associated to M.

Homogenous and symmetric spaces

$(g, p) \longmapsto g \cdot p \in M$ **transitive action** of G on M

$H = \left\{g \in G : g \cdot p_0 = p_0\right\}$ **isotropy group** at $p_0 \in M$

G/H is a **homogenous space** isomorphic to M

Reductive homogenous spaces and symmetric spaces

The homogenous space G/H is **reductive** if there exists a subspace $\mathfrak{p}$ of $\mathfrak{g}$ such that

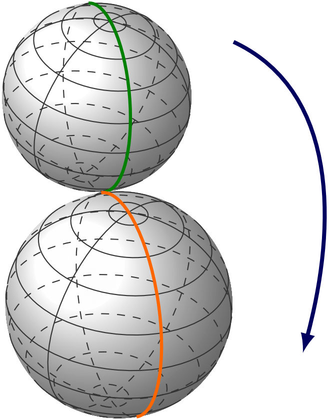
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p},$$

where $\mathfrak{h}$ is the Lie algebra of H and $\left[\mathfrak{h}, \mathfrak{p}\right] \subseteq \mathfrak{p}$.
Moreover, if $\left[\mathfrak{p}, \mathfrak{p}\right] \subseteq \mathfrak{h}$, G/H is a **homogenous symmetric space**.

$\pi_{p_0} : g \longmapsto g \cdot p_0 \in M$ **Riemannian submersion**

$$T_{p_0}M \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{p}$$

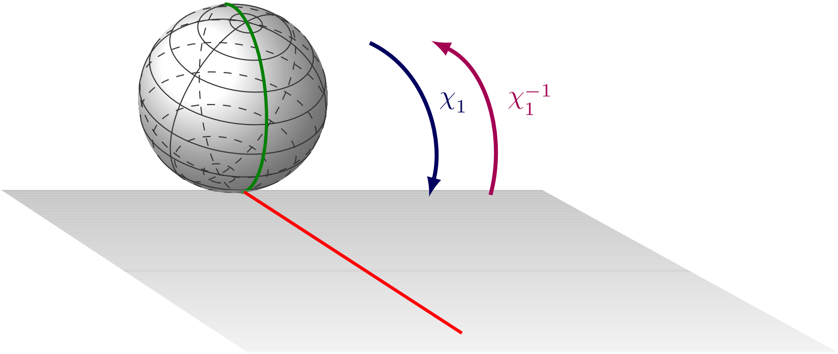
Properties of the Rolling maps



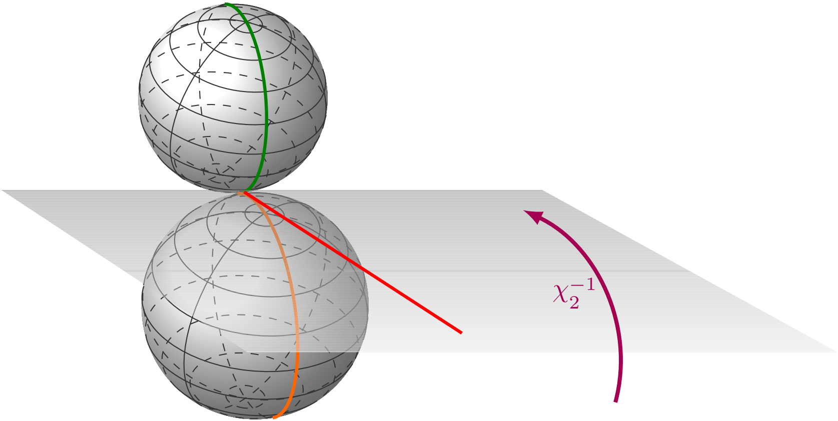
$\chi = ?$

Properties of the Rolling maps

Symmetry

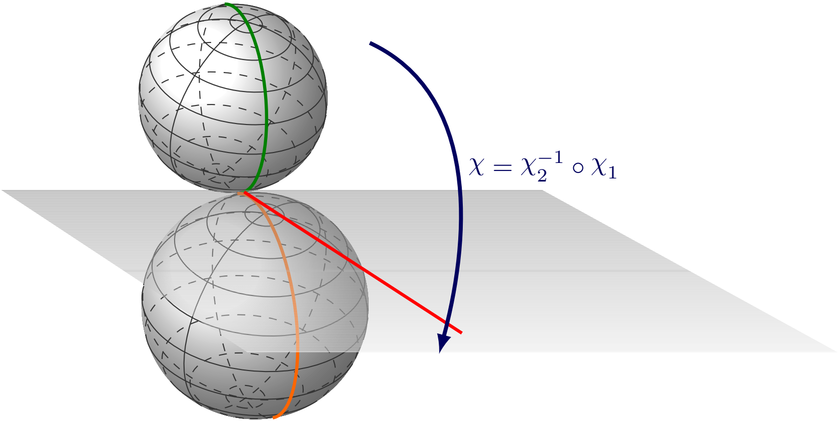


Transitivity



Properties of the Rolling maps

Transitivity



Rolling symmetric homogeneous spaces

Assume that $\widetilde{M}$ is the Euclidean space $\mathbb{R}^m$

M **Riemannian symmetric homogeneous space** G/H embedded in $\mathbb{R}^m$ (H isotropy group at p_0)

$\widetilde{G} = SO(m) \ltimes \mathbb{R}^m$ **group of isometries** of $\mathbb{R}^m$

$t \mapsto \chi(t) = \big(R(t), s(t)\big) \in SO(m) \ltimes \mathbb{R}^m$ **rolling map** of M upon $T_{p_0}^{\mathrm{aff}} M$

Action of $\chi = (\mathrm{R}, \mathrm{s})$ on M and on $\mathrm{T_p M}$

$$\chi(t) \cdot p = R(t) \cdot p + s(t)$$

$$d_p\chi(t) \cdot V = R(t) \cdot V$$

Main results

M **Riemannian symmetric homogeneous space** G/H

Proposition

If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ ($\mathfrak{h}$ is the Lie algebra of H), then

$$\mathfrak{h}\left(T_{p_0}M\right) \subseteq T_{p_0}M \qquad \mathfrak{h}\left(T_{p_0}^{\perp}M\right) \subseteq T_{p_0}^{\perp}M$$

and

$$\mathfrak{p}\left(T_{p_0}M\right) \subseteq T_{p_0}^{\perp}M \qquad \mathfrak{p}\left(T_{p_0}^{\perp}M\right) \subseteq T_{p_0}M$$

Theorem

If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and χ is the rolling map of $M = G/H$ upon $T_{p_0}^{\mathrm{aff}}M$, then $d\left(\dot{\chi}\chi^{-1}\right)$ is an element of $\mathfrak{p}$.

The $\mathbf{n}-$sphere $\mathbf{S^n}$

$\mathbf{S^n \simeq SO(n+1)/SO(n)}$ is a **homogeneous symmetric space** under the natural action of $SO(n+1)$: $\quad (\Theta, p) \mapsto \Theta p$

The **isotropy group** at $p_0 = (0, \ldots, 0, -1)$:

$$H = \left\{ \left[\begin{array}{c|c} \Theta & 0 \\ \hline 0 & 1 \end{array} \right] \ : \ \Theta \in SO(n) \right\} \simeq SO(n)$$

$\mathfrak{so}(n+1) = \mathfrak{h} \oplus \mathfrak{p}$

$$\mathfrak{h} = \left\{ \left[\begin{array}{c|c} \Omega & 0 \\ \hline 0 & 0 \end{array} \right] \ : \ \Omega \in \mathfrak{so}(n) \right\} \qquad \mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & v \\ \hline -v^T & 0 \end{array} \right] \ : \ v \in \mathbb{R}^n \right\}$$

It is well known that the **kinematic equations** for the rolling sphere are

$$\begin{cases} \dot{s}(t) = A(t)p_0 \\ \dot{R}(t) = -A(t)R(t) \end{cases},$$

where

$$A(t) = \left[\begin{array}{c|c} 0 & v(t) \\ \hline -v(t)^T & 0 \end{array}\right].$$

[Jurdjevic (1997)]

Rolling the $\mathrm{n}-$sphere S^{n}

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$$\begin{cases} \dot{s}(t) = A(t)p_0 \\ \dot{R}(t) = -A(t)R(t) \end{cases},$$

where

$$A(t) = \left[\begin{array}{c|c} 0 & v(t) \\ \hline -v(t)^T & 0 \end{array}\right].$$

Then,

$$d\big(\dot{\chi}\chi^{-1}\big) = \dot{R}R^{-1} = -A \in \mathfrak{p}.$$

The Lorentzian sphere $S^{n,1}$

For $J = \mathrm{diag}(1, \dots, 1, -1)$, consider the pseudo-orthogonal Lie group

$$SO(n,1) = \left\{ X \in \mathbb{R}^{(n+1)\times(n+1)} \;:\; X^T J X = J \wedge \det(X) = 1 \right\}$$

$\mathbf{S^{n,1} \simeq SO(n,1)/SO(n-1,1)}$ is a **homogeneous symmetric space** under the natural action of $SO(n,1)$: $(X, p) \mapsto Xp$

The **isotropy group** at $p_0 = (1, 0, \dots, 0)$:

$$H = \left\{ \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Theta \end{array} \right] \;:\; \Theta \in SO(n-1,1) \right\} \simeq SO(n-1,1)$$

$\mathfrak{so}(n,1) = \mathfrak{h} \oplus \mathfrak{p}$

$$\mathfrak{h} = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Omega \end{array} \right] : \Omega \in \mathfrak{so}(n-1,1) \right\} \qquad \mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & -v^T \\ \hline v & 0 \end{array} \right] J \;: v \in \mathbb{R}^n \right\}$$

Rolling the Lorentzian sphere

The **kinematic equations** for the rolling Lorentzian sphere are

$$\begin{cases} \dot{s}(t) = u(t) \\ \dot{R}(t) = \left(p_0\, u(t)^T - u(t)\, p_0^T\right) J R(t) \end{cases},$$

where $u(t) \in T_{p_0} S^{n,1}$.

[Jurdjevic & Zimmerman (2008)] and [Korolko & Silva Leite (2011)]

Rolling the Lorentzian sphere

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where $u(t) \in T_{p_0} S^{n,1}$.

Then,

$$d\left(\dot{\chi}\chi^{-1}\right) = \dot{R}R^{-1} = \left(p_0\, u^T - u\, p_0^T\right) J \in \mathfrak{p}.$$

The Grassmann manifold

$$\mathbf{G_{n,k}} = \left\{ P \in \mathfrak{s}(n) \;:\; P^2 = P \,\wedge\, \mathsf{rank}(P) = k \right\}$$

$\mathbf{G_{n,k}} \simeq \mathbf{SO(n)/SO(k) \times SO(n-k)}$ is a **homogeneous symmetric space** under the action of $SO(n)$: $\quad (X, P) \mapsto XPX^T$

The **isotropy group** at $P_0 = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$:

$$H = \left\{ \left[\begin{array}{c|c} X_1 & 0 \\ \hline 0 & X_2 \end{array}\right] \;:\; X_1 \in SO(k),\; X_2 \in SO(n-k) \right\} \simeq SO(k) \times SO(n-k)$$

$\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{p}$

$$\mathfrak{h} = \left\{ \left[\begin{array}{c|c} \Omega_1 & 0 \\ \hline 0 & \Omega_2 \end{array}\right] : \Omega_1 \in \mathfrak{so}(k),\, \Omega_2 \in \mathfrak{so}(n-k) \right\} \quad \mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & V \\ \hline -V^T & 0 \end{array}\right] : V \in \mathbb{R}^{k \times (n-k)} \right\}$$

Rolling the Grassmann manifold

The **kinematic equations** for the rolling of the Grassmann manifold are

$$\dot{X}(t) = \left[\begin{array}{c|c} 0 & \Psi(t) \\ \hline \Psi(t)^T & 0 \end{array}\right],$$
$$\dot{R}(t) = \left[\begin{array}{c|c} 0 & \Psi(t) \\ \hline -\Psi(t)^T & 0 \end{array}\right] R(t)$$

where $\Psi(t) \in \mathbb{R}^{k\times(n-k)}$.

[Hüper & Silva Leite (2007)]

The **kinematic equations** for the rolling of the Grassmann manifold are

$$\dot{X}(t) = \left[\begin{array}{c|c} 0 & \Psi(t) \\ \hline \Psi(t)^T & 0 \end{array}\right] \qquad ,$$
$$\dot{R}(t) = \left[\begin{array}{c|c} 0 & \Psi(t) \\ \hline -\Psi(t)^T & 0 \end{array}\right] R(t)$$

where $\Psi(t) \in \mathbb{R}^{k\times(n-k)}$.

Then,

$$d\big(\dot{\chi}\chi^{-1}\big) = \mathsf{ad}_{\dot{R}R^{-1}} \in \mathfrak{p}.$$

Conjecture

If $M = G/H$ is a Riemannian symmetric homogenous space and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ ($\mathfrak{h}$ is the Lie algebra of H), then

$$\mathfrak{p}\big(T_{p_0}M\big) \subseteq T^{\perp}_{p_0}M, \qquad \mathfrak{p}\big(T^{\perp}_{p_0}M\big) \subseteq T_{p_0}M$$

and therefore

$$d\big(\dot{\chi}\chi^{-1}\big) \in \mathfrak{p},$$

even when the **ambient manifold $\widetilde{M}$ is non-Euclidean**.