Entropy minimizing curves

Application to automated flight path design

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Problem Statement

Flight path planning

- Traffic is expected to double by 2050 ;
- In future systems, trajectories will be negotiated and optimized well before the flights start ;
- But humans will be in the loop : generated flight plans must comply with operational constraints ;

Muti-agent systems

- A promising approach to address the planning problem ;
- Does not end up with a human friendly traffic !
- Idea : start with the proposed solution and rebuild a route network from it.

A curve optimization problem

An entropy criterion

- Route networks and currently made of straight segments connecting beacons ;
- May be viewed as a maximally concentrated spatial density distribution ;
- Minimizing the entropy with such a density will intuitively yield a flight path system close to what is expected.

Problem modeling

Density associated with a curve system

- A classical measure : counting the number of aircraft in each bin of a spatial grid and averaging over time ;
- Suffers from a severe flaw : aircraft with low velocity will over-contribute ;
- May be corrected by enforcing invariance under re-parametrization of curves ;
- Combined with a non-parametric kernel estimate to yield :

$$\tilde{d} \colon x \mapsto \frac{\sum_{i=1}^{N} \int_0^1 K\left(\|x-\gamma_i(t)\|\right)\|\gamma_i'(t)\|dt}{\sum_{i=1}^{N} \int_\Omega \int_0^1 K\left(\|x-\gamma_i(t)\|\right)\|\gamma_i'(t)\|dtdx} \qquad (1)$$

Problem modeling II

The entropy criterion

- Kernel K is normalized over the domain Ω so as to have a unit integral ;
- Density is directly related to lengths $l_i, i = 1 \dots n$ of curves $\gamma_i, i = 1 \dots N$:

$$\tilde{d} \colon x \mapsto \frac{\sum_{i=1}^{N} \int_0^1 K\left(\|x - \gamma_i(t)\|\right) \|\gamma_i'(t)\| dt}{\sum_{i=1}^{N} l_i} \tag{2}$$

- Associated entropy is :

$$E(\gamma_1, \dots, \gamma_N) = -\int_\Omega \tilde{d}(x) \log\left(\tilde{d}(x)\right) dx \tag{3}$$

Optimal curve displacement field

Entropy variation

- $\tilde{d}$ has integral 1 over the domain Ω ;
- It implies that :

$$-\frac{\partial}{\partial \gamma_j} E(\gamma_1, \ldots, \gamma_N)(\epsilon) = \int_\Omega \frac{\partial \tilde{d}(x)}{\partial \gamma_j}(\epsilon) \log\left(\tilde{d}(x)\right) dx \quad (4)$$

where ϵ is an admissible variation of curve γ_i.

- The denominator in the expression of $\tilde{d}$ has derivative :

$$\int_{[0,1]} \left\langle \frac{\gamma_j'(t)}{\|\gamma_j'(t)\|}, \epsilon'(t) \right\rangle dt = -\int_{[0,1]} \left\langle \left(\frac{\gamma_j''(t)}{\|\gamma_j'(t)\|} \right)_{\mathcal{N}}, \epsilon \right\rangle dt \quad (5)$$

Optimal curve displacement field

Entropy variation

- The numerator of $\tilde{d}$ has derivative :

$$\int_{[0,1]} \left\langle \left(\frac{\gamma_j(t) - x}{\|\gamma_j(t) - x\|} \right)_{\mathcal{N}}, \epsilon \right\rangle K' \left(\|\gamma_j(t) - x\| \right) \|\gamma_j'(t)\| dt \quad (6)$$

$$- \int_{[0,1]} \left\langle \left(\frac{\gamma_j''(t)}{\|\gamma_j'(t)\|} \right)_{\mathcal{N}}, \epsilon \right\rangle K \left(\|\gamma_j(t) - x\| \right) dt \quad (7)$$

Optimal curve displacement field II

Normal move

- Final expression yield a displacement field normal to the curve :

$$\left(\int_\Omega \left(\frac{\gamma_j(t) - x}{\|\gamma_j(t) - x\|} \right)_{\mathcal{N}} K' \left(\|\gamma_j(t) - x\| \right) \log \tilde{d}(x) dx \|\gamma_j'(t)\| \right. \tag{8}$$

$$- \left(\int_\Omega K \left(\|\gamma_j(t) - x\| \right) \log \tilde{d}(x)) dx \right) \left(\frac{\gamma_j''(t)}{\|\gamma_j'(t)\|} \right)_{\mathcal{N}} \tag{9}$$

$$\left. + \left(\int_\Omega \tilde{d}(x) \log(\tilde{d}(x)) dx \right) \left(\frac{\gamma_j''(t)}{\|\gamma_j'(t)\|} \right)_{\mathcal{N}} \right) \Big/ \sum_{i=1}^{n} l_i \tag{10}$$

Implementation

A gradient algorithm

- The move is based on a tangent vector in the tangent space to $\mathbf{Imm}([0,1], \mathbb{R}^3)/\mathbf{Diff}^+([0,1))$;
- It is not directly implementable on a computer;
- A simple, landmark based approach with evenly spaced points was used;
- A compactly supported kernel (epanechnikov) was selected : it allows the computation of density $\tilde{d}$ on GPUs as a texture operation that is very fast.

A output from the multi-agent system

Integration in the complete system

- Route building from initially conflicting trajectories :

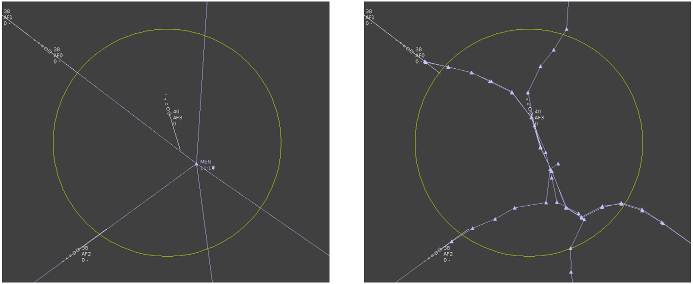

Figure – Initial flight plans and final ones

Conclusion and future work

An integrated algorithm

- Entropy minimizer is now a part of the overall route design system ;
- Only a simple post-processing is necessary to output a usable airways network ;
- The complete algorithm is being ported to GPU.

Future work : take the headings into account

- The behavior is not completely satisfactory when routes are converging in opposite directions ;
- An improved version will make use of entropy of a distribution in a Lie group (publication in progress).