Transformations and Coupling Relations for Affine Connections

James Tao (Harvard University, Cambridge MA)
Jun Zhang (University of Michigan, Ann Arbor MI)

Oct 29, 2015

Outline

1. Transformation of affine connections (with torsion)
   - *h*-conjugation: by a two-form h
   - gauge transform: by an operator L
   - additive translation: by a (1,2)-tensor K
2. Commutative relations and "commutativity prisms"
   - keeping track of "torsion" as going through the transformations
3. Transformations that preserve Codazzi coupling (g, ∇)
   - more general than "conformal-projective transformation"?

Outline

1. Transformation of affine connections (with torsion)
    - h-conjugation: by a two-form h
    - gauge transform: by an operator L
    - additive translation: by a (1,2)-tensor K
2. Commutative relations and "commutativity prisms"
    - keeping track of "torsion" as going through the transformations
3. Transformations that preserve Codazzi coupling (g, ∇)
    - more general than "conformal-projective transformation"?

Outline

1. Transformation of affine connections (with torsion)
   - *h*-conjugation: by a two-form *h*
   - gauge transform: by an operator *L*
   - additive translation: by a (1,2)-tensor *K*
2. Commutative relations and "commutativity prisms"
   - keeping track of "torsion" as going through the transformations
3. Transformations that preserve Codazzi coupling (g, ∇)
   - more general than "conformal-projective transformation"?

Statistical manifold and Codazzi coupling

On a differentiable manifold $\mathfrak{M}$, one *independently* prescribes:

1. a pseudo-Riemannian metric g;
2. an affine connection ∇.

Codazzi coupling of g and ∇

The pair (g, ∇) is said to be Codazzi-coupled if

$$(\nabla_Z g)(X, Y) = (\nabla_X g)(Z, Y).$$

This notion is a generalization of Levi-Civita coupling (i.e., parallelism of g with respect to ∇). It can be shown that (∇, g) is Codazzi-coupled $\longleftrightarrow$ ∇ and ∇^* have same torsion.

Statistical manifold: definition

A manifold $(\mathcal{M}, g, \nabla)$ where (i) ∇ is torsion-free and (ii) (g, ∇) is Codazzi-coupled.

Statistical manifold and Codazzi coupling

On a differentiable manifold $\mathfrak{M}$, one *independently* prescribes:

1. a pseudo-Riemannian metric g;
2. an affine connection ∇.

Codazzi coupling of g and ∇

The pair (g, ∇) is said to be Codazzi-coupled if

$$(\nabla_Z g)(X, Y) = (\nabla_X g)(Z, Y).$$

This notion is a generalization of Levi-Civita coupling (i.e., parallelism of g with respect to ∇). It can be shown that (∇, g) is Codazzi-coupled $\longleftrightarrow$ ∇ and ∇^* have same torsion.

Statistical manifold: definition

A manifold $(\mathcal{M}, g, \nabla)$ where (i) ∇ is torsion-free and (ii) (g, ∇) is Codazzi-coupled.

Statistical manifold and Codazzi coupling

On a differentiable manifold $\mathfrak{M}$, one *independently* prescribes:

1. a pseudo-Riemannian metric g;
2. an affine connection ∇.

Codazzi coupling of g and ∇

The pair (g, ∇) is said to be Codazzi-coupled if

$$(\nabla_Z g)(X, Y) = (\nabla_X g)(Z, Y).$$

This notion is a generalization of Levi-Civita coupling (i.e., parallelism of g with respect to ∇). It can be shown that (∇, g) is Codazzi-coupled $\longleftrightarrow$ ∇ and ∇^* have same torsion.

Statistical manifold: definition

A manifold $(\mathcal{M}, g, \nabla)$ where (i) ∇ is torsion-free and (ii) (g, ∇) is Codazzi-coupled.

Statistical manifold and Codazzi coupling

On a differentiable manifold $\mathfrak{M}$, one *independently* prescribes:

1. a pseudo-Riemannian metric g;
2. an affine connection ∇.

Codazzi coupling of g and ∇

The pair (g, ∇) is said to be Codazzi-coupled if

$$(\nabla_Z g)(X, Y) = (\nabla_X g)(Z, Y).$$

This notion is a generalization of Levi-Civita coupling (i.e., parallelism of g with respect to ∇). It can be shown that (∇, g) is Codazzi-coupled $\longleftrightarrow$ ∇ and ∇^* have same torsion.

Statistical manifold: definition

A manifold $(\mathcal{M}, g, \nabla)$ where (i) ∇ is torsion-free and (ii) (g, ∇) is Codazzi-coupled.

Conjugate connection

g-conjugation of a connection

Given any (g, ∇), conjugate connection ∇^* can be defined:

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y).$$

It can be verified that (i) ∇^* is indeed a connection and (ii) the $*$ action on ∇ is involutive: $(\nabla^*)^* = \nabla$.

Defining a connection by conjugacy with a non-degenerate two-form h:

- can be done unambiguously only when h is symmetric or skew-symmetric;
- otherwise "left conjugate" and "right conjugate", in reference to the slot $h(\cdot, \cdot)$, will not be the same.

Conjugate connection

g-conjugation of a connection

Given any (g, ∇), conjugate connection ∇^* can be defined:

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y).$$

It can be verified that (i) ∇^* is indeed a connection and (ii) the $*$ action on ∇ is involutive: $(\nabla^*)^* = \nabla$.

Defining a connection by conjugacy with a non-degenerate two-form h:

- can be done unambiguously only when h is symmetric or skew-symmetric;
- otherwise "left conjugate" and "right conjugate", in reference to the slot $h(\cdot,\cdot)$, will not be the same.

Gauge transformation of connection

Let L denote $T\mathfrak{M}$ isomorphism. The gauge transformation of ∇ by L, denoted $L(\nabla)$, is defined as (for vector fields X, Y):

$$(L(\nabla))_X Y = L^{-1}(\nabla_X(LY)).$$

(L, ∇) is said to be Codazzi-coupled if

$$(\nabla_X L)Y = (\nabla_Y L)X,$$

where

$$(\nabla_X L)Y \equiv \nabla_X(LY) - L(\nabla_X Y).$$

Proposition (Schwenk-Schellschmidt and Simon, 2009)

Let ∇ be an affine connection, and L be a tangent bundle isomorphism. Then the following are equivalent:

1. *(∇, L) is Codazzi-coupled.*
2. *∇ and $L(\nabla)$ have equal torsions.*
3. *$(L(\nabla), L^{-1})$ is Codazzi-coupled.*

Gauge transformation of connection

Let L denote $T\mathfrak{M}$ isomorphism. The gauge transformation of ∇ by L, denoted $L(\nabla)$, is defined as (for vector fields X, Y):

$$(L(\nabla))_X Y = L^{-1}(\nabla_X(LY)).$$

(L, ∇) is said to be Codazzi-coupled if

$$(\nabla_X L)Y = (\nabla_Y L)X,$$

where

$$(\nabla_X L)Y \equiv \nabla_X(LY) - L(\nabla_X Y).$$

Proposition (Schwenk-Schellschmidt and Simon, 2009)

Let ∇ be an affine connection, and L be a tangent bundle isomorphism. Then the following are equivalent:

1. *(∇, L) is Codazzi-coupled.*
2. *∇ and $L(\nabla)$ have equal torsions.*
3. *$(L(\nabla), L^{-1})$ is Codazzi-coupled.*

Gauge transformation of connection

Let L denote $T\mathfrak{M}$ isomorphism. The gauge transformation of ∇ by L, denoted $L(\nabla)$, is defined as (for vector fields X, Y):

$$(L(\nabla))_X Y = L^{-1}(\nabla_X(LY)).$$

(L, ∇) is said to be Codazzi-coupled if

$$(\nabla_X L)Y = (\nabla_Y L)X,$$

where

$$(\nabla_X L)Y \equiv \nabla_X(LY) - L(\nabla_X Y).$$

Proposition (Schwenk-Schellschmidt and Simon, 2009)

Let ∇ be an affine connection, and L be a tangent bundle isomorphism. Then the following are equivalent:

1. (∇, L) *is Codazzi-coupled.*
2. ∇ *and* $L(\nabla)$ *have equal torsions.*
3. $(L(\nabla), L^{-1})$ *is Codazzi-coupled.*

Linking g-conjugation with L-gauge transform

We proved the following characterization theorem for g-conjugation of a connection in terms of any L:

Characterization Theorem

Let ∇ be a connection and ∇^* its conjugate connection w.r.t. a metric g. Denote $\omega(X, Y) = g(LX, Y)$ for arbitrary $T\mathfrak{M}$ isomorphism L. Then $\nabla\omega = 0$ if and only if

$$L(\nabla^*) = \nabla.$$

Explicitly written:

$$\nabla^*_Z X = \nabla_Z X + L(\nabla_Z L^{-1})X.$$

Proof used the identify (for any invertible operator L):

$$C_h(X, Y, Z) = C_g(L(X), Y, Z) + g((\nabla_Z L)X, Y),$$

where $C(X, Y, Z) \equiv (\nabla_Z g)(X, Y)$, $h(X, Y) \equiv g(L(X), Y)$.

Translation of a connection by K-tensor

Translation by a (1,2)-tensor: $\nabla_X Y \to \nabla_X Y + K(X,Y)$. It is torsion-preserving iff K is symmetric: $K(X,Y) = K(Y,X)$.

Examples of K-translations

(i) $\mathrm{P}^\vee(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(X)Y$, $\mathrm{P}^\vee$-*transformation*;

(ii) $\mathrm{P}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X$, P-*transformation*;

(iii) $\mathrm{Proj}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X + \tau(X)Y$, called *projective transformation*, always torsion-preserving;

(iv) $\mathrm{D}(h,V) : \nabla_X Y \mapsto \nabla_X Y - h(Y,X)V$, called "dual-projective transformation", torsion-preserving when h symmetric.

Here τ is an arbitrary one-form, h is a non-degenerate two-form, X, Y, V are all vector fields.

Translation of a connection by K-tensor

Translation by a (1,2)-tensor: $\nabla_X Y \to \nabla_X Y + K(X,Y)$. It is torsion-preserving iff K is symmetric: $K(X,Y) = K(Y,X)$.

Examples of K-translations

(i) $\mathrm{P}^\vee(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(X)Y$, $\mathrm{P}^\vee$-*transformation*;

(ii) $\mathrm{P}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X$, P-*transformation*;

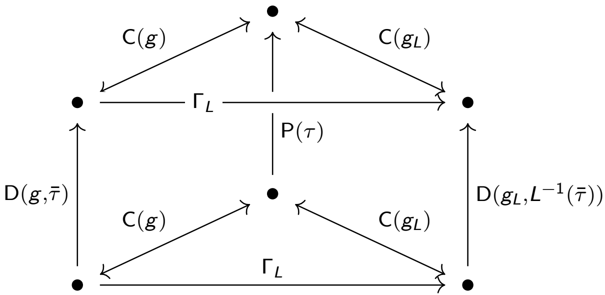
(iii) $\mathrm{Proj}(\tau) : \nabla_X Y \mapsto \nabla_X Y + \tau(Y)X + \tau(X)Y$, called *projective transformation*, always torsion-preserving;

(iv) $\mathrm{D}(h,V) : \nabla_X Y \mapsto \nabla_X Y - h(Y,X)V$, called "dual-projective transformation", torsion-preserving when h symmetric.

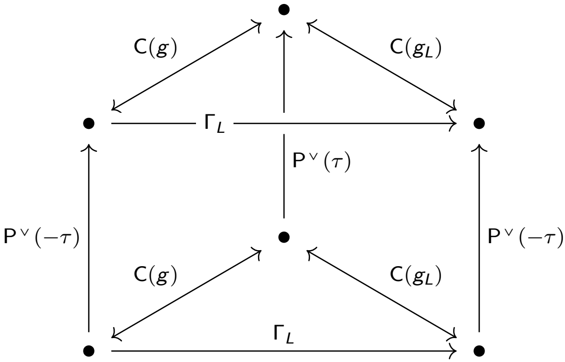
Here τ is an arbitrary one-form, h is a non-degenerate two-form, X, Y, V are all vector fields.

Interactions of h-conjugation, L-gauge, K-translation

Let g, L, τ be as above. Let g_L denote $g(L\cdot, \cdot)$, Γ_L denote L-gauge transformation, $\mathrm{C}(g)$ denote conjugation w.r.t. g, and $\bar{\tau}$ be the vector field such that $g(X, \bar{\tau}) = \tau(X)$.



Interactions of h-conjugation, L-gauge, K-translation



Conformal-projective transformation (CPT)

Conformal-projective transformation (CPT) is defined (Kurose, 2002) as, for any smooth functions ψ and ϕ,

$$g(X,Y) \mapsto e^{\psi+\phi} g(X,Y)$$
$$\nabla_X Y \mapsto \nabla_X Y - g(X,Y)\,\mathrm{grad}_g\,\psi + X(\phi)Y + Y(\phi)X$$

CPT include, as special cases,

- *projective transformation* of ∇
- *conformal transformation* of g and Levi-Civita connection
- *dual-projective transformation* of ∇, given (g, ∇)
- *Codazzi transform* of g and ∇
- α-*conformal transformation* of g and ∇

It is known that CPT preserves Codazzi coupling of (g, ∇). We wonder whether it can be further generalized while preserving Codazzi structure.

Conformal-projective transformation (CPT)

Conformal-projective transformation (CPT) is defined (Kurose, 2002) as, for any smooth functions ψ and ϕ,

$$g(X,Y) \mapsto e^{\psi+\phi} g(X,Y)$$
$$\nabla_X Y \mapsto \nabla_X Y - g(X,Y)\,\mathrm{grad}_g\,\psi + X(\phi)Y + Y(\phi)X$$

CPT include, as special cases,

- *projective transformation* of ∇
- *conformal transformation* of g and Levi-Civita connection
- *dual-projective transformation* of ∇, given (g,∇)
- *Codazzi transform* of g and ∇
- α*-conformal transformation* of g and ∇

It is known that CPT preserves Codazzi coupling of (g,∇). We wonder whether it can be further generalized while preserving Codazzi structure.

Conformal-projective transformation (CPT)

Conformal-projective transformation (CPT) is defined (Kurose, 2002) as, for any smooth functions ψ and ϕ,

$$g(X, Y) \mapsto e^{\psi+\phi} g(X, Y)$$
$$\nabla_X Y \mapsto \nabla_X Y - g(X, Y)\, \mathrm{grad}_g\, \psi + X(\phi)Y + Y(\phi)X$$

CPT include, as special cases,

- *projective transformation* of ∇
- *conformal transformation* of g and Levi-Civita connection
- *dual-projective transformation* of ∇, given (g, ∇)
- *Codazzi transform* of g and ∇
- α*-conformal transformation* of g and ∇

It is known that CPT preserves Codazzi coupling of (g, ∇).
We wonder whether it can be further generalized while preserving Codazzi structure.

Conformal-projective transformation (CPT)

Conformal-projective transformation (CPT) is defined (Kurose, 2002) as, for any smooth functions ψ and ϕ,

$$g(X,Y) \mapsto e^{\psi+\phi} g(X,Y)$$
$$\nabla_X Y \mapsto \nabla_X Y - g(X,Y)\,\mathrm{grad}_g\,\psi + X(\phi)Y + Y(\phi)X$$

CPT include, as special cases,

- *projective transformation* of ∇
- *conformal transformation* of g and Levi-Civita connection
- *dual-projective transformation* of ∇, given (g,∇)
- *Codazzi transform* of g and ∇
- α*-conformal transformation* of g and ∇

It is known that CPT preserves Codazzi coupling of (g,∇). We wonder whether it can be further generalized while preserving Codazzi structure.

$CP(V, W, L)$ preserving Codazzi Structure

Generalized conformal-projective transformation $CP(V, W, L)$

Let V and W be vector fields, and L an invertible operator. $\mathrm{CP}(V, W, L)$ consists of an L-perturbation of the metric g along with a torsion-preserving transformation $\mathrm{D}(g, W)\mathrm{Proj}(\tilde{V})$ of the connection, where $\tilde{V}$ is the one-form given by $\tilde{V}(X) := g(V, X)$ for any vector field X.

Proposition. (Assuming $\dim \mathfrak{M} \geqslant 4$)

$\mathrm{CP}(V, W, L)$ preserves Codazzi pairs $\{\nabla, g\}$ if and only if $L = e^f$ for some smooth function f, and $V + W = \mathrm{grad}_g f$.

Take $\tilde{V}$ to be an arbitrary one-form, not necessarily closed, and $\tilde{W} := df - \tilde{V}$ for some fixed smooth function f. CPT results when $f = \phi + \psi$, in which case $df = d\phi + d\psi$ is a natural decomposition.

$CP(V, W, L)$ preserving Codazzi Structure

Generalized conformal-projective transformation $CP(V, W, L)$

Let V and W be vector fields, and L an invertible operator. $\mathrm{CP}(V, W, L)$ consists of an L-perturbation of the metric g along with a torsion-preserving transformation $\mathrm{D}(g, W)\mathrm{Proj}(\tilde{V})$ of the connection, where $\tilde{V}$ is the one-form given by $\tilde{V}(X) := g(V, X)$ for any vector field X.

Proposition. (Assuming $\dim \mathfrak{M} \geqslant 4$)

$\mathrm{CP}(V, W, L)$ preserves Codazzi pairs $\{\nabla, g\}$ if and only if $L = e^f$ for some smooth function f, and $V + W = \mathrm{grad}_g f$.

Take $\tilde{V}$ to be an arbitrary one-form, not necessarily closed, and $\tilde{W} := df - \tilde{V}$ for some fixed smooth function f. CPT results when $f = \phi + \psi$, in which case $df = d\phi + d\psi$ is a natural decomposition.

Recent development (Teng Fei and Jun Zhang)

Let L be J (almost compatible structure) or K (almost para-complex structure): $J^2 = -id$; $K^2 = id$.
A compatible triple (g, ω, L) satisfies:

1. $g(LX, Y) + g(X, LY) = 0$;
2. $\omega(LX, Y) = \omega(X, LY)$;
3. $\omega(X, Y) = g(LX, Y)$;

A manifold $\mathfrak{M}$ is called:

1. *symplectic* if there exists a symplectic (skew-symmetric + non-degenerate) form ω that is closed: $d\omega = 0$;
2. *Fedosov* if (i) $\mathfrak{M}$ is symplectic and (ii) there exists a torsion-free connection parallel to $\omega : \nabla\omega = 0$;
3. *(para)Kähler* if (i) $\mathfrak{M}$ is symplectic and (ii) there exists an integrable L compatible with $\omega : \omega(X, LY) = \omega(LX, Y)$.

Recent development (Teng Fei and Jun Zhang)

Let L be J (almost compatible structure) or K (almost para-complex structure): $J^2 = -id; K^2 = id$.

A compatible triple (g, ω, L) satisfies:

1. $g(LX, Y) + g(X, LY) = 0$;
2. $\omega(LX, Y) = \omega(X, LY)$;
3. $\omega(X, Y) = g(LX, Y)$;

A manifold $\mathfrak{M}$ is called:

1. *symplectic* if there exists a symplectic (skew-symmetric + non-degenerate) form ω that is closed: $d\omega = 0$;
2. *Fedosov* if (i) $\mathfrak{M}$ is symplectic and (ii) there exists a torsion-free connection parallel to ω : $\nabla\omega = 0$;
3. *(para)Kähler* if (i) $\mathfrak{M}$ is symplectic and (ii) there exists an integrable L compatible with ω : $\omega(X, LY) = \omega(LX, Y)$.

Codazzi Structure and (Para)-Kähler Structure

Main Theorem

Let ∇ be a torsion-free connection on M, and L denote either J (almost complex) or K (almost para-complex) operator on TM. Then, for the following three statements, any two imply the third:

1. ∇ is Codazzi-coupled with g;
2. ∇ is Codazzi-coupled with L;
3. $\nabla\omega = 0$.

As a result, M becomes a Kähler or para-Kähler manifold.

In other words, Codazzi coupling of (∇, L) turns a statistical manifold *or* Fedosov manifold into a (para-)Kähler manifold, which is then *both* statistical *and* symplectic.

THANK YOU FOR ATTENTION!!

Tao, J. and Zhang, J. (2015). Transformation and coupling relations for affine connections. Proceedings of GSI 2015. Springer.
Fei, T. and Zhang, J, (in preparation). Interaction of Codazzi structur and (para)-Kahler structure.