The Pontryagin Forms of Hessian Manifolds

J. Armstrong S.Amari

October 27, 2015

Summary

Question

Given a Riemannian metric g, under what circumstances is it locally a Hessian metric?

Question

When can we locally find a function f and coordinates x such that $g_{ij} = \partial_i \partial_j f$?

Answer (Partial)

In dimension 2 all analytic metrics g are Hessian. In dimensions 3 the general metric is not Hessian. In dimensions $\geqslant 4$ there are even restrictions on the curvature tensor of g — in particular the Pontrjagin forms vanish.

Solving unusual partial differential equations

Question

Given a symmetric g, when can we locally find a function f and coordinates x such that $g_{ij} = (\partial_i f)(\partial_j f)$?

Answer

Only if g lies in the n dimensional subspace $\operatorname{Im}\phi \subset S^2 T$ where

$$\phi : T \to S^2 T \qquad \text{by } \phi(x) = x \odot x.$$

Sometimes we can't find a solution even at a point.

Question

Given a one form η, when can we locally find a function f such that $\mathrm{d}f = \eta$.

Answer

Since $\mathrm{dd}f = 0$ we must have $\mathrm{d}\eta = 0$ at x. *Sometimes we can find a solution at a point, but can't extend it even to first order around x.*

Generalizing

- Let E and F be vector bundles and let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator.
- $D : J_k(E) \to F$ where J_k is the bundle of k jets.
- Define $D_1 : J_{k+1}(E) \to J_1(F)$ to be the first *prolongation*. This is the operator which maps a section e to the one jet of $j_1(De)$.
- Define $D_i : J_{k+i}(E) \to J_i(F)$ to be the i-th prolongation $e \to j_i(e)$

We can only hope to solve the differential equation $De = f$ if we can find an algebraic solution to every equation

$$D_i e = j_i(f)$$

at the point x.
Applying the fact that derivatives commute may yield obstructions to the existence of solutions to a differential equation even locally.

Dimension counting

- The dimension of the space of k-jets of 1 functions of n real variables is:

$$\dim J_k := \sum_{i=0}^{k+2} \dim(S^i T) = \sum_{i=0}^{k} \binom{n+i-1}{i}.$$

The reason for this is that derivatives commute. Note this fact is also encoded in the statement $\mathrm{d}\mathrm{d}f = 0$.

The counting argument

- We wish to solve
$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f = g_{ij}.$$
which is a second order equation for f and coords x. So input is $n+1$ functions of n variables.
- Dimension of space of $(k+2)$ jets of f and x
$$d_k^1 = \dim J_{k+2}(x, f) = \sum_{i=0}^{k+2} (n+1)\binom{n+i-1}{i}.$$
- Dimension of space of k jets of g:
$$d_k^2 = \dim J_k(g) = \sum_{i=0}^{k} \frac{n(n+1)}{2}\binom{n+i-1}{i}.$$
- If $n > 2$ d_k^1 grows more slowly than d_k^2. So most metrics are not Hessian metrics.

Informal version

- A Riemannian metric depends on $\frac{n(n+1)}{2}$ functions of n variables.
- A Hessian metric depends on $n+1$ functions of n variables.
- "Therefore" if $n > 2$ there are more Riemannian metrics than Hessian metrics.
- Note: this computation is suggestive but slightly wrong because we've ignored the diffeomorphism group. It would suggest that in dimension 1 there are more Hessian metrics than Riemannian metrics!

Curvature

Reminder:

- Hessian metrics locally correspond to g-dually flat structures, and vice versa.
- g-dually flat means $\overline{\nabla}$ is flat and it's dual w.r.t. g $\overline{\nabla}^*$ is flat.

$$g(\nabla_Z X, Y) = g(X, \nabla^*_Z Y).$$

Proposition

Let (M, g) be a Riemannian manifold. Let ∇ denote the Levi–Civita connection and let $\overline{\nabla} = \nabla + A$ be a g-dually flat connection. Then

(i) The tensor A_{ijk} lies in $S^3 T^*$. We shall call it the S^3-*tensor* of $\overline{\nabla}$.

(ii) The S^3-tensor determines the Riemann curvature tensor as follows:

$$R_{ijkl} = -g^{ab} A_{ika} A_{jlb} + g^{ab} A_{ila} A_{jkb}.$$

Proof

- $\overline{\nabla}$ is torsion free implies $A \in S^2 T^* \otimes T$
- Using metric to identify $T^* and T$, both $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion free implies $A \in S^3 T^*$
- $\overline{R} = 0$. But by definition:

$$\overline{R}_{XY} Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]} Z$$

Expanding in terms of Levi–Civita:

$$\overline{R}_{XY} Z = R_{XY} Z + 2(\nabla_{[X} A)_{Y]} Z + 2A_{[X} A_{Y]} Z$$

Curvature symmetries tell us (using g to identify T and T^*):

$$R \in \Lambda^2 T \otimes \Lambda^2 T$$

On the other hand:

$$(\nabla_{[.} A)_{.]} \in \Lambda^2 T \otimes S^2 T$$

Projecting the equation onto $\Lambda^2 T \otimes \Lambda^2 T$ gives the desired result.

Curvature obstruction

Define a quadratic equivariant map ρ from $S^3 T^* \longrightarrow \Lambda^2 T^* \otimes \Lambda^2 T^*$ by:

$$\rho(A_{ijk}) = -g^{ab} A_{ika} A_{jlb} + g^{ab} A_{ila} A_{jkb}$$

If g is a Hessian metric R lies in image of ρ.

Corollary

In dimension $\geqslant 5$, ρ is not onto. Therefore there condition $R \in \operatorname{Im} \rho$ is an obstruction to a metric being a Hessian metric.

Proof.

$$\dim \mathcal{R} = \dim(\text{Space of algebraic curvature tensors}) = \frac{1}{12} n^2 (n^2 - 1)$$

$$\dim(S^3 T) = \frac{1}{6} n(1+n)(2+n)$$

The former is strictly greater than the latter if $n \geqslant 5$ □

Dimension 4

Numerical observation: ρ is not onto in dimension 4 even though $\dim \mathcal{R} = \dim(S^3 T^*) = 20$.

Proof.

Pick a random $A \in S^3 T^*$ and compute rank of $(\rho *)_A$, the differential of ρ at A. It is 18 whereas the space of algebraic curvature tensors is 20 dimensional. (Proof with probability 1) □

Question

What are the conditions on the curvature tensor for it to lie in the image of ρ?

What does this question mean?

- This is an *implicitization* question. $\operatorname{Im} \rho$ is given parametrically by the map ρ. We want implicit equations on the curvature tensor that define $\operatorname{Im} \rho$.
- This is a real algebraic geometry question and so we should expect inequalities for our implicit equations. (e.g. $\operatorname{Im} x^2 = \{y : y \geqslant 0\}$)
- Complexify the vector spaces to get a complex algebraic geometry where we expect equalities for our implicit equations. This is how we choose to interpret the question.
- Gröbner basis algorithms allow us to solve the latter problem in principle (for fixed n) but not in practice (doubly exponential time is common).
- Algorithms do exist for the real algebraic geometry problem too, but they're even less practical.

Strategy

- Space of algebraic curvature tensors $\mathcal{R}$ is associated to a representation of $SO(n)$.
- Decompose $\mathcal{R}$ into irreducible components under $SO(n)$
- Any invariant linear condition on $\mathcal{R}$ can be expressed as a linear combination of these irreducibles.
- Decompose $S^2\mathcal{R} \oplus \mathcal{R}$ into irreducibles. Any invariant quadratic condition on $\mathcal{R}$ can be expressed as a linear combination of these irreducibles. etc.
- If we have m irreducible components $\rho_1(R)$, $\rho_2(R)$, ..., $\rho_m(R)$. Choose $m+1$ random tensors A and solve the equation
$$\sum_i \alpha_i \rho_i(R) = 0$$
for α_i. (In fact we only need to check linear combinations over isomorphic components)
- This is feasible in dimension 4. Representation theory of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is simple. is simple

Hessian curvature tensors in dimension 4

Theorem

The space of possible curvature tensors for a Hessian 4-manifold is 18 dimensional. In particular the curvature tensor must satisfy the identities:

$$\alpha(R_{ija}{}^{b} R_{klb}{}^{a}) = 0$$

$$\alpha(R_{iajb} R_{k}{}^{b}{}_{cd} R_{l}{}^{dac} - 2 R_{iajb} R_{kc}{}^{a}{}_{d} R_{l}{}^{dbc}) = 0$$

where α denotes antisymmetrization of the i, j, k and l indices.

Proof.

Using a symbolic algebra package, write the general tensor in $S^3 T^*$ with respect to an orthonormal basis in terms of its 20 components. Compute the curvature tensor using ρ. One can then directly check the above identities. □

- Both expressions define 4-forms on a general Riemannian manifold. The first is a well-known 4-form. It defines the first Pontrjagin class of the manifold.

Pontrjagin forms

- The Gauss–Bonnet formula gives an important link between curvature and topology. In this case the integral of scalar curvature is related to the Euler class.
- The theory of *characteristic classes* generalizes this.
  - To a complex vector bundle V over a manifold M one can associate topological invariants, the Chern classes $c_i(V) \in H^{2i}(M)$.
  - The Pontrjagin classes of a real vector bundle $V^{\mathbb{R}}$ are defined to be the Chern classes of the complexification $p_i(V^{\mathbb{R}}) \in H^{4i}(M)$.
  - The Pontrjagin classes of a manifold are defined to be the Pontrjagin classes of its tangent bundle.
  - It is possible to find explicit representatives for the De Rham cohomology classes of a bundle by computing appropriate polynomial expressions if a curvature tensor for the bundle.
  - We call these explicit representatives *Pontrjagin forms*.

Relationship between Pontrjagin forms and curvature

Theorem

For each p, the form $Q_p(R)$ defined by:

$$Q^p_{i_1 i_2 \dots i_{2p}} = \sum_{\sigma \in S_{2p}} \operatorname{sgn}(\sigma) R_{i_{\sigma(1)} i_{\sigma(2)} a_1}{}^{a_2} R_{i_{\sigma(3)} i_{\sigma(4)} a_2}{}^{a_3} R_{i_{\sigma(5)} i_{\sigma(6)} a_3}{}^{a_4} \dots R_{i_{\sigma(2p-1)} i_{\sigma(2p)} a_p}{}^{a_1}$$

is closed. The Pontrjagin forms can all be written as algebraic expressions in these $Q_p(R)$ using the ring structure of Λ^* and vice-versa.

This is a standard result from the theory of characteristic classes.

Main result

Theorem

The forms $Q_p(R)$ vanish on Hessian manifolds, hence the Pontrjagin forms vanish on Hessian manifolds.

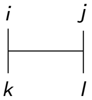
Corollary

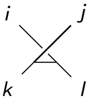
If a manifold M admits a metric that is everywhere locally Hessian then its Pontrjagin classes all vanish.

Note that we're being clear to distinguish this from the case of a manifold which is globally dually flat, where the vanishing of the Pontrjagin classes is a trivially corollary of the existence of flat connections.

Graphical notation

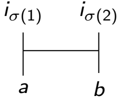
$$\rho(A_{ijk}) = -g^{ab}A_{ika}A_{jlb} + g^{ab}A_{ila}A_{jkb}$$

$R_{ijkl} = -$ 

$+$ 

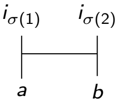
.

- Trivalent graph
- Each vertex represents the tensor A
- Connecting vertices represents contraction with the metric
- Picture naturally incorporates symmetries of A

$R_{i_1 i_2 ab} = \sum_{\sigma \in S_2} -\operatorname{sgn}(\sigma)$ 

.

Proof

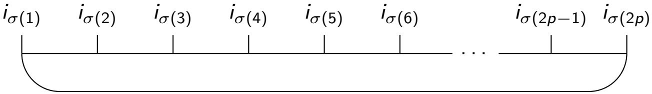
$$R_{i_1 i_2 ab} = \sum_{\sigma \in S_2} - \operatorname{sgn}(\sigma)$$ 

.

By definition:

$$Q^p_{i_1 i_2 \ldots i_{2p}} = \sum_{\sigma \in S_{2p}} \operatorname{sgn}(\sigma) R_{i_{\sigma(1)} i_{\sigma(2)} a_1}{}^{a_2} R_{i_{\sigma(3)} i_{\sigma(4)} a_2}{}^{a_3} R_{i_{\sigma(5)} i_{\sigma(6)} a_3}{}^{a_4} \ldots R_{i_{\sigma(2p-1)} i_{\sigma(2p)} a_p}{}^{a_1}$$

We can replace each R with an H:

$$Q^p_{i_1 i_2 \ldots i_{2p}} = (-1)^p \sum_{\sigma \in S_{2p}} \operatorname{sgn}(\sigma)$$ 

Since the cycle $1 \to 2 \to 3 \ldots \to 2p \to 1$ is an odd permutation, one sees that $Q^p = 0$.

Summary

- In dimension 2 all metrics are locally Hessian (Use Cartan–Kähler theory. Proved independently by Robert Bryant)
- In dimensions $\geqslant 3$ not all metrics are locally Hessian
- In dimensions $\geqslant 4$ there are conditions on the curvature
- In dimension 4 we have identified two conditions explicitly. These are necessary conditions and, working over the complex numbers, they characterize $\operatorname{Im} \rho$.
- In dimension $n \geqslant 4$ we have identified a number of explicit curvature conditions in terms of the Pontrjagin forms. Dimension counting tells us that other curvature conditions exist, but we do not know them explicitly.