Stochastic PDE projection on manifolds: Assumed-Density and Galerkin Filters

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Damiano Brigo
Dept. of Mathematics, Imperial College, London
www.damianobrigo.it

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Joint work with *John Armstrong*
Dept. of Mathematics, King's College, London

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Full paper to appear in MCSS, see also arXiv.org

Spaces of probability densities

Consider a parametric family of probability densities

$$S = \{p(\cdot,\theta), \theta \in \Theta \subset \mathbb{R}^m\}, \quad S^{1/2} = \{\sqrt{p(\cdot,\theta)}, \theta \in \Theta \subset \mathbb{R}^m\}.$$

If S (or $S^{1/2}$) is a subset of a function space having an L^2 structure ($\Rightarrow$ inner product, norm & metric), then we may ask whether

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is a Chart of a m-dim manifold (?) S ($S^{1/2}$). The topology & differential structure in the chart is the L^2 structure, but two possibilities:

$$S: \quad d_2(p_1, p_2) = \|p_1 - p_2\| \quad \text{(L2 direct distance)}, \quad p_{1,2} \in L^2$$

$$S^{1/2}: \quad d_H(\sqrt{p_1}, \sqrt{p_2}) = \|\sqrt{p_1} - \sqrt{p_2}\| \quad \text{(Hellinger distance)}, \quad p_{1,2} \in L^1$$

where $\|\cdot\|$ is the norm of Hilbert space L^2.

Tangent vectors, metrics and projection

If $\varphi : \theta \mapsto p(\cdot, \theta)$ ($\theta \mapsto \sqrt{p(\cdot, \theta)}$ resp.) is the inverse of a chart then

$$\{\frac{\partial \varphi(\cdot, \theta)}{\partial \theta_1}, \cdots, \frac{\partial \varphi(\cdot, \theta)}{\partial \theta_m}\}$$

are linearly independent $L^2(\lambda)$ vector that span Tangent Space at θ.

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The inner product of 2 basis elements is defined (L^2 structure)

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$$\left(\langle \frac{\partial \sqrt{p}}{\partial \theta_i} \frac{\partial \sqrt{p}}{\partial \theta_j} \rangle = \frac{1}{4} \int \frac{1}{p(x,\theta)} \frac{\partial p(x,\theta)}{\partial \theta_i} \frac{\partial p(x,\theta)}{\partial \theta_j} dx = \frac{1}{4} g_{ij}(\theta) \right) .$$

$\gamma(\theta)$: direct L2 matrix (d_2); $g(\theta)$: famous Fisher-Rao matrix (d_H)

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d_2 ort. projection: $\Pi_\theta^\gamma[v] = \sum_{i=1}^m [\sum_{j=1}^m \gamma^{ij}(\theta) \langle v, \frac{\partial p(\cdot,\theta)}{\partial \theta_j} \rangle] \frac{\partial p(\cdot,\theta)}{\partial \theta_i}$

(d_H proj. analogous inserting $\sqrt{\cdot}$ and replacing γ with g)

The nonlinear filtering problem for diffusion signals

$$\begin{aligned} dX_t &= f_t(X_t)\,dt + \sigma_t(X_t)\,dW_t, \quad X_0, \quad \text{(signal)} \\ dY_t &= b_t(X_t)\,dt + dV_t, \quad Y_0 = 0 \;\; \text{(noisy observation)} \end{aligned} \tag{1}$$

These are Itô SDE's. We use both Itô and Stratonovich (Str) SDE's. Str SDE's are necessary to deal with manifolds, since second order Itô terms not clear in terms of manifolds [16], *although we are working on a direct projection of Ito equations with good optimality properties (John Armstrong)*

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The **nonlinear filtering problem** consists in finding the conditional probability distribution π_t of the state X_t given the observations up to time t, i.e. $\pi_t(dx) := P[X_t \in dx \mid \mathcal{Y}_t]$, where $\mathcal{Y}_t := \sigma(Y_s\,,\, 0 \le s \le t)$. Assume π_t has a density p_t: then p_t satisfies the Str SPDE:

The nonlinear filtering problem for diffusion signals

$$dp_t = \mathcal{L}_t^* \, p_t \, dt - \frac{1}{2} p_t \, [|b_t|^2 - E_{p_t}\{|b_t|^2\}] \, dt + \sum_{k=1}^{d} p_t \, [b_t^k - E_{p_t}\{b_t^k\}] \circ dY_t^k \, .$$

with the forward operator $\mathcal{L}_t^* \phi = - \sum_{i=1}^n \frac{\partial}{\partial x_i} [f_t^i \, \phi] + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} [a_t^{ij} \, \phi]$

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With Ito calculus we would have terms $\frac{\partial^2 p(\cdot, \theta)}{\partial \theta_i \partial \theta_j} d\langle \theta_i, \theta_j \rangle$ (not tang vec)

Projection filter in the metrics h (L2) and g (Fisher)

$$d\theta_t^i = \left[\sum_{j=1}^m \gamma^{ij}(\theta_t) \int \mathcal{L}_t^* \, p(x,\theta_t) \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} dx - \sum_{j=1}^m \gamma^{ij}(\theta_t) \int \frac{1}{2}|b_t(x)|^2 \, \frac{\partial p}{\partial \theta_j} \, dx\right] dt$$

$$+ \sum_{k=1}^d [\sum_{j=1}^m \gamma^{ij}(\theta_t) \int b_t^k(x) \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, dx] \circ dY_t^k \ , \quad \theta_0^i \ .$$

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The above is the projected equation in d_2 metric and Π^γ.
Instead, using the Hellinger distance & the Fisher metric with projection Π^g

$$d\theta_t^i = \left[\sum_{j=1}^m g^{ij}(\theta_t) \int \frac{\mathcal{L}_t^* \, p(x,\theta_t)}{p(x,\theta_t)} \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, dx - \sum_{j=1}^m g^{ij}(\theta_t) \int \frac{1}{2} \, |b_t(x)|^2 \frac{\partial p}{\partial \theta_j} dx \right] dt$$

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Choosing the family/manifold: Exponential

In past literature and in several papers in Bernoulli, IEEE Automatic Control etc, B. Hanzon and LeGland have developed a theory for the projection filter using the Fisher metric g and exponential families $p(x, \theta) := \exp[\theta^T c(x) - \psi(\theta)]$. Good combination:

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- One can define both a local and global filtering error through d_H
- Alternative coordinates, expectation param., $\eta = E_\theta[c] = \partial_\theta \psi(\theta)$.
- Projection filter in η coincides with classical approx filter: assumed density filter (based on generalized "moment matching")

Mixture families

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- The answer is affirmative, and this is the *mixture family*.

We define a *simple mixture family* as follows. Given $m+1$ fixed squared integrable probability densities $\underline{q} = [q_1, q_2, \dots, q_{m+1}]^T$, define

$$\hat{\theta}(\theta) := [\theta_1, \theta_2, \dots, \theta_m, 1 - \theta_1 - \theta_2 - \dots - \theta_m]^T$$

for all $\theta \in \mathbb{R}^m$. We write $\hat{\theta}$ instead of $\hat{\theta}(\theta)$. Mixture family (simplex):

$$S^M(\underline{q}) = \{\hat{\theta}(\theta)^T \underline{q}, \;\; \theta_i \geq 0 \text{ for all } i, \;\; \theta_1 + \dots + \theta_m < 1\}$$

Mixture families

If we consider the L^2 / $\gamma(\theta)$ distance, the metric $\gamma(\theta)$ itself and the related projection become very simple. Indeed,

$$\frac{\partial p(\cdot,\theta)}{\partial \theta_i} = q_i - q_{m+1} \text{ and } \gamma_{ij}(\theta) = \int (q_i(x) - q_m(x))(q_j(x) - q_m(x))dx$$

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(NO inline numeric integr). The L^2 metric *does not depend on the specific point* θ *of the manifold*. The same holds for the tangent space at $p(\cdot,\theta)$, which is given by

$$\text{span}\{q_1 - q_{m+1}, q_2 - q_{m+1}, \cdots, q_m - q_{m+1}\}$$

Also the L^2 projection becomes particularly simple.

Mixture Projection Filter

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- See the full paper for the details. Summing up:

Family $\rightarrow$ Metric $\downarrow$	Exponential	Basic Mixture
Hellinger d_H Fisher $g(\theta)$	Good $\sim$ADF $\approx$ local moment matching	Nothing special
Direct L^2 d_2 matrix $\gamma(\theta)$	Nothing special	Good ($\sim$Galerkin)

Mixture Projection Filter

- However, despite the simplicity above, the mixture family has an important drawback: for all θ, filter mean is constrained

$$\min_i \text{mean of } q_i \leq \text{mean of } p(\cdot, \theta) \leq \max_i \text{mean of } q_i$$

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- Specifically, we consider a mixture of GAUSSIAN DENSITIES with MEANS AND VARIANCES in each component not fixed. For example for a mixture of two Gaussians we have 5 parameters.

$$\theta p_{\mathcal{N}(\mu_1, v_1)}(x) + (1-\theta) p_{\mathcal{N}(\mu_2, v_2)}(x), \quad \text{param.} \quad \theta, \mu_1, v_1, \mu_2, v_2$$

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- We are now going to illustrate the Gaussian mixture projection filter (GMPF) in a fundamental example.

The quadratic sensor

Consider the quadratic sensor

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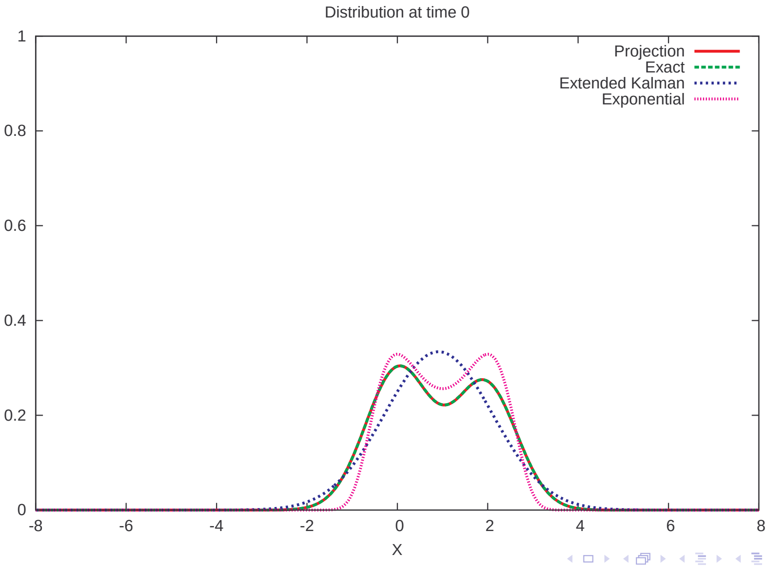
- Once it seems likely that the state has moved past the origin, the distribution will become nearly symmetrical
- We expect a bimodal distribution
- $\theta p_{\mathcal{N}(\mu_1,v_1)}(x) + (1-\theta) p_{\mathcal{N}(\mu_2,v_2)}(x)$ (red)
- vs $e^{\theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4 - \psi(\theta)}$ (pink)
- vs EKF ($\mathcal{N}$) (blue)
- vs exact (green, finite diff. method, grid 1000 state & 5000 time)

Simulation for the Quadratic Sensor



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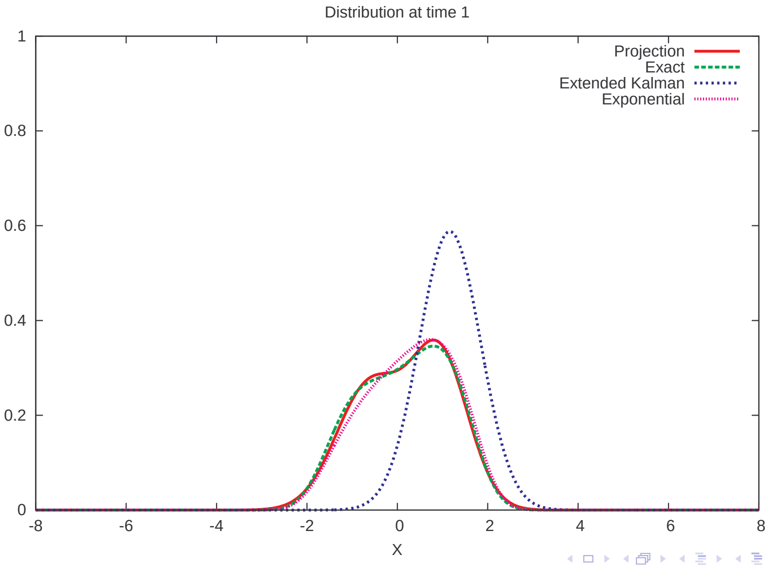

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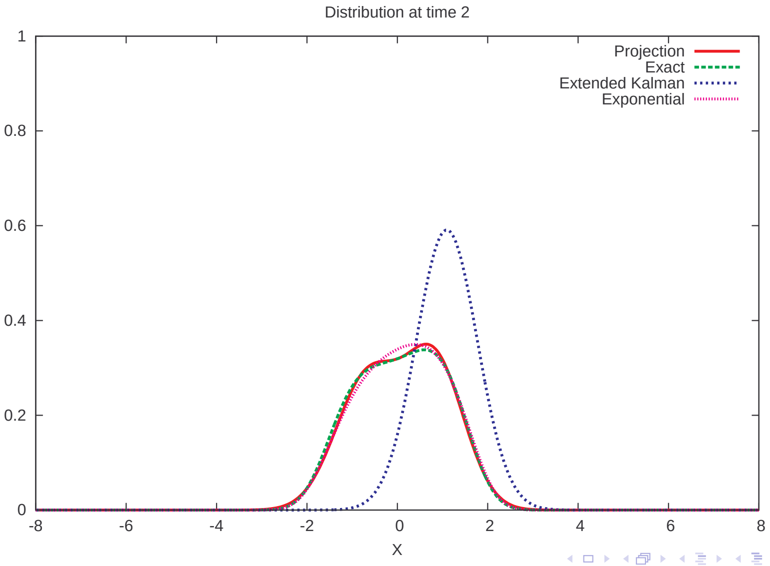

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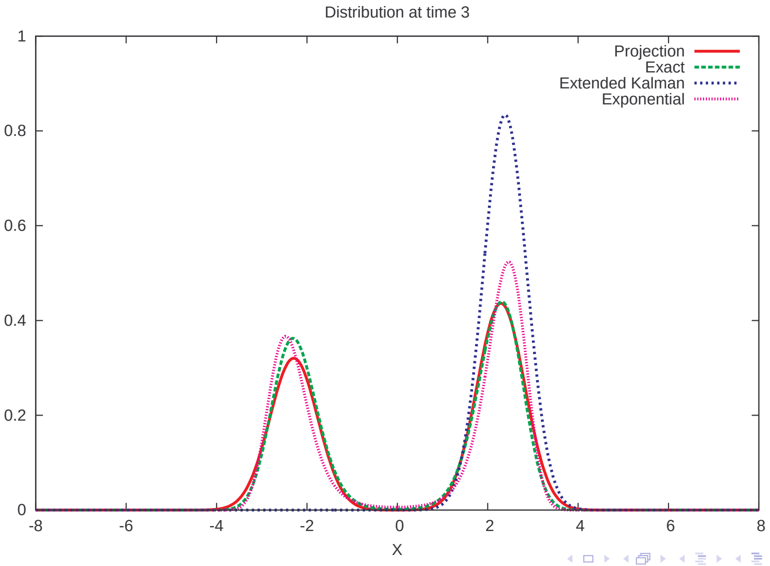

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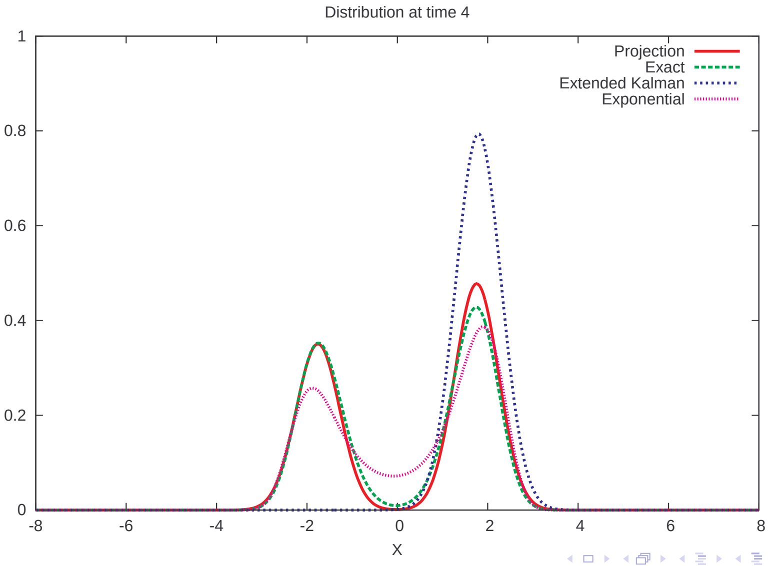

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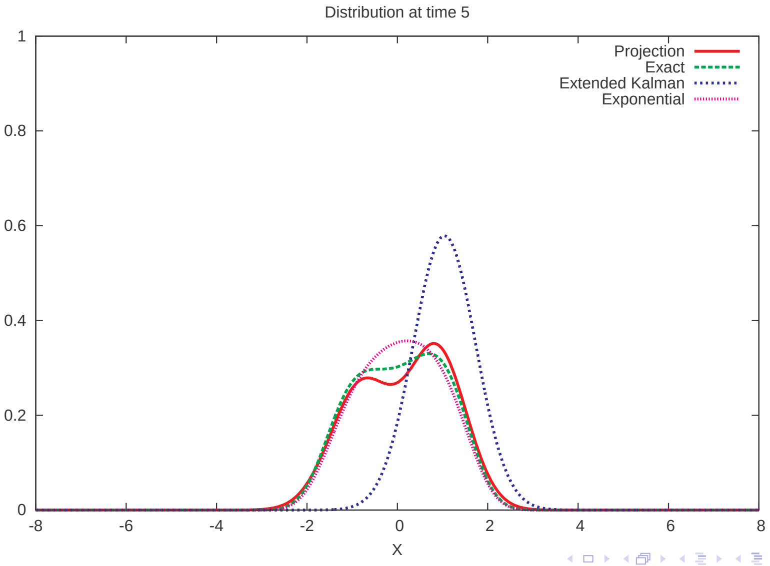

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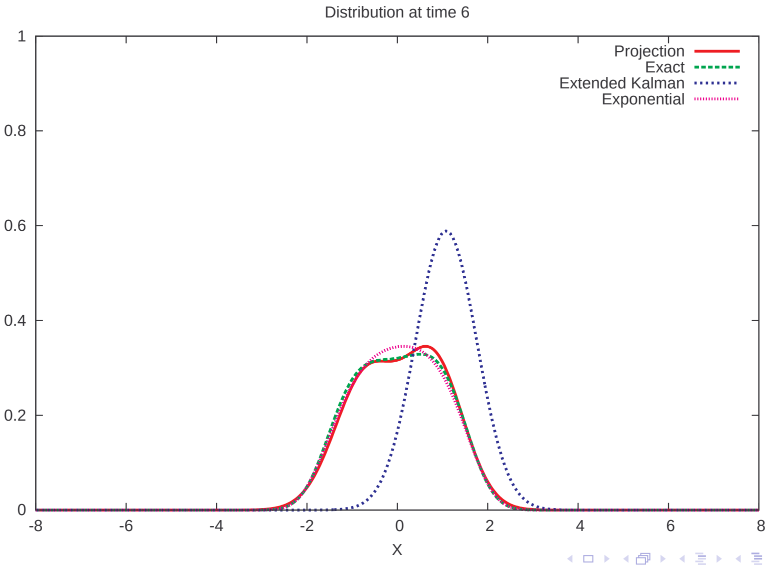

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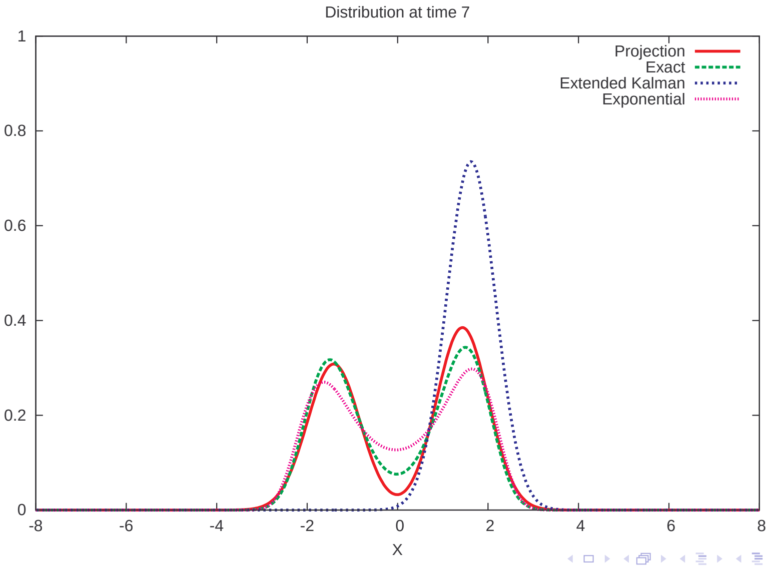

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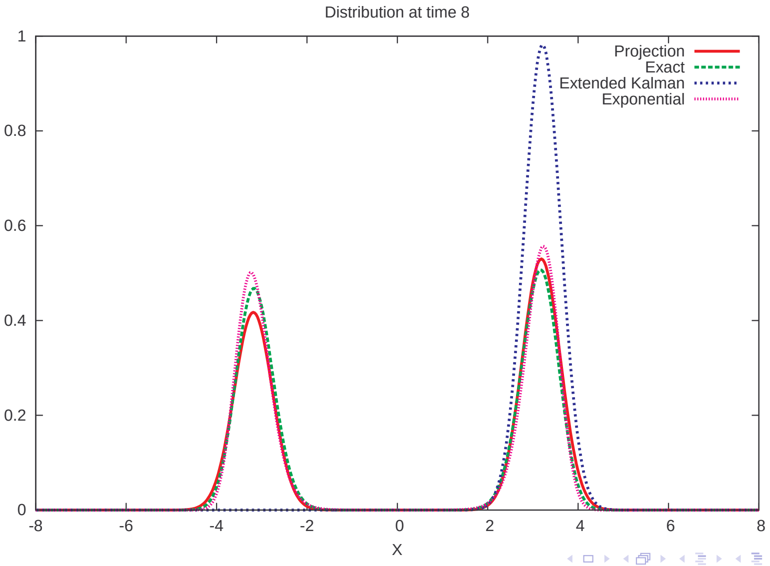

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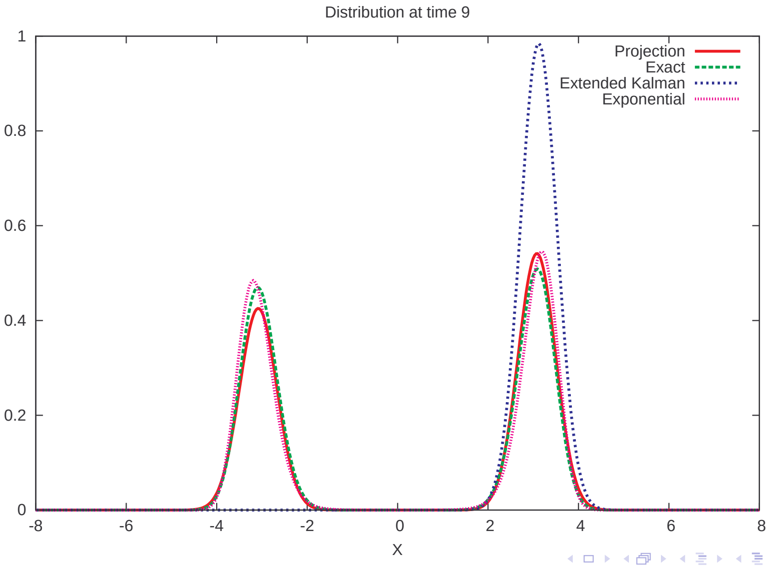

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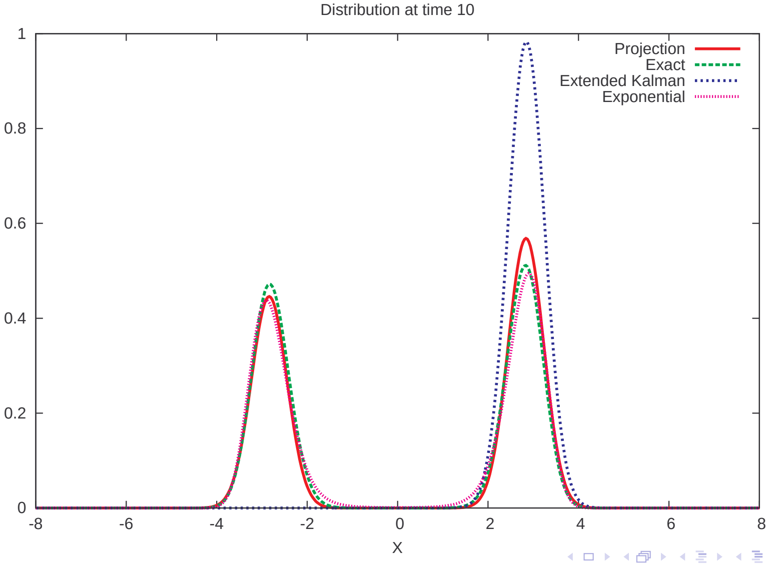
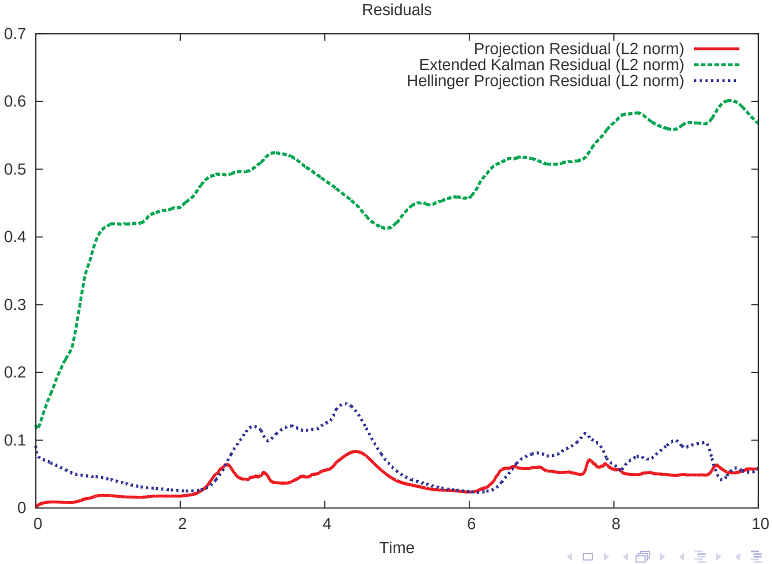


Comparing local approximation errors (L^2 residuals) ε_t

$$\varepsilon_t^2 = \int (p_{\text{exact},t}(x) - p_{\text{approx},t}(x))^2 dx$$

- $p_{\text{approx},t}(x)$: three possible choices.
- $\theta p_{\mathcal{N}(\mu_1, v_1)}(x) + (1-\theta) p_{\mathcal{N}(\mu_2, v_2)}(x)$ (red)
- vs $e^{\theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4 - \psi(\theta)}$ (blue)
- vs EKF ($\mathcal{N}$) (green)

L^2 residuals for the quadratic sensor



Comparing local approx errors (Prokhorov residuals) ε_t

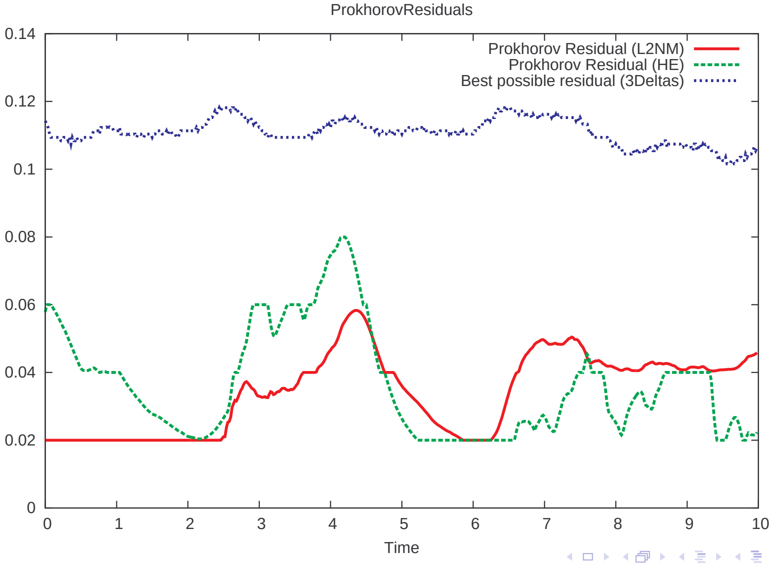
$$\varepsilon_t = \inf\{\epsilon : F_{\text{exact},t}(x-\epsilon) - \epsilon \le F_{\text{approx},t}(x) \le F_{\text{exact},t}(x+\epsilon) + \epsilon \ \ \forall x\}$$

with F the CDF of p's.

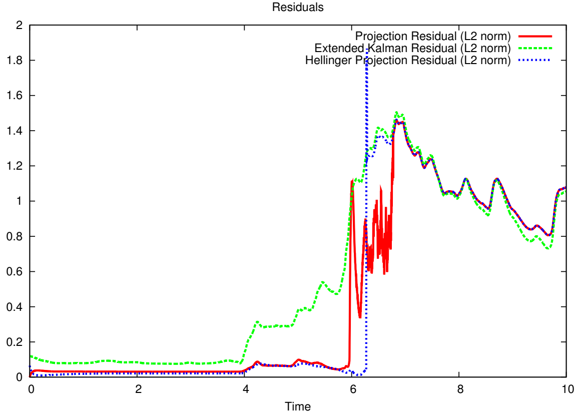
Levy-Prokhorov metric works well with singular densities like particles where L2 metric not ideal.

- $\theta p_{\mathcal{N}(\mu_1,v_1)}(x) + (1-\theta) p_{\mathcal{N}(\mu_2,v_2)}(x)$ (red)
- vs $e^{\theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4 - \psi(\theta)}$ (green)
- vs best three particles (blue)

Lévy residuals for the quadratic sensor



Cubic sensors



- Qualitatively similar results up to a stopping time

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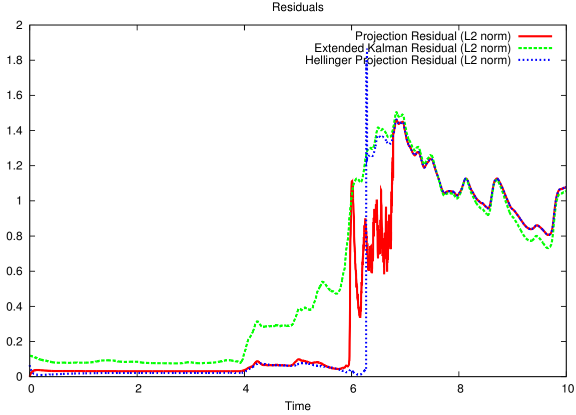

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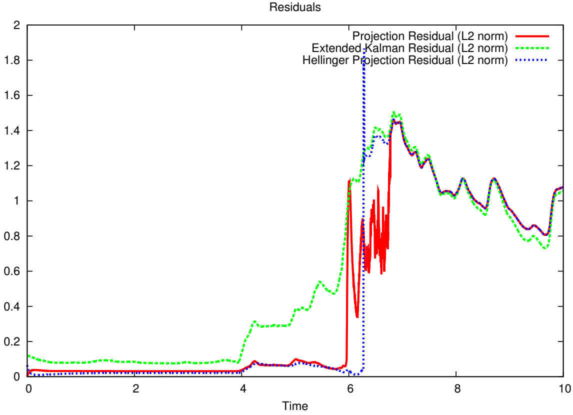

- Qualitatively similar results up to a stopping time
- As one approaches the boundary γ_{ij} becomes singular
- The solution is to dynamically change the parameterization and even the dimension of the manifold.

Conclusions

- Approximate finite-dimensional filtering by rigorous projection on a chosen manifold of densities

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- Further investigation: convergence, more on *optimality*?
- Optimality: introducing new projections (forthcoming J. Armstrong)

Thanks

With thanks to the organizing committee.

Thank you for your attention.

Questions and comments welcome

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