Group Theoretical Study on Geodesics for the Elliptical Models

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Overview

1. Eriksen's construction of geodesics on normal model
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2. Reconsideration of Eriksen's argument
   - Embedding $N_n \hookrightarrow \mathrm{Sym}^+_{n+1}(\mathbb{R})$
3. Geodesic equation on Elliptical model
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Eriksen's construction of geodesics on normal model

Let $\mathrm{Sym}_n^+(\mathbb{R})$ be the set of n-dimensional positive-definite matrices.
The **normal model** $N_n = (M, ds^2)$ is a Riemannian manifold defined by

$$M = \left\{(\mu, \Sigma) \in \mathbb{R}^n \times \mathrm{Sym}_n^+(\mathbb{R})\right\},$$

$$ds^2 = ({}^t d\mu)\Sigma^{-1}(d\mu) + \frac{1}{2}\mathrm{tr}((\Sigma^{-1}d\Sigma)^2).$$

The **geodesic equation** on N_n is

$$\begin{cases} \ddot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\mu} = 0, \\ \ddot{\Sigma} + \dot{\mu}{}^t\dot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\Sigma} = 0. \end{cases} \tag{1}$$

The solution of this geodesic equation has been obtained by Eriksen.

Theorem ([Eriksen 1987])

For any $x \in \mathbb{R}^n, B \in \mathrm{Sym}_n(\mathbb{R})$*, define a matrix exponential* $\Lambda(t)$ *by*

$$\Lambda(t) = \begin{pmatrix} \Delta & \delta & \Phi \\ {}^t\delta & \epsilon & {}^t\gamma \\ {}^t\Phi & \gamma & \Gamma \end{pmatrix} := \exp(-tA), \quad A := \begin{pmatrix} B & x & 0 \\ {}^tx & 0 & -{}^tx \\ 0 & -x & -B \end{pmatrix} \in \mathrm{Mat}_{2n+1}. \tag{2}$$

Then, the curve $(\mu(t), \Sigma(t)) := (-\Delta^{-1}\delta, \Delta^{-1})$ *is the geodesic on* N_n *satisfiying the initial condition*

$$(\mu(0), \Sigma(0)) = (0, I_n), \quad (\dot{\mu}(0), \dot{\Sigma}(0)) = (x, B).$$

(proof)

We see that by the definition, $(\mu(t), \Sigma(t))$ satisfies the geodesic equation.

Problem

1. Explain Eriksen's theorem, to clarify the relation between the normal model and symmetric spaces.
2. Extend Eriksen's theorem to the elliptical model.

$\mathrm{Sym}^{+}_{n+1}(\mathbb{R})$

Notice that the positive-definite symmetric matrices $\mathrm{Sym}^{+}_{n+1}(\mathbb{R})$ is a symmetric space by

$$\begin{array}{ccc} G/K & \simeq & \mathrm{Sym}^{+}_{n+1}(\mathbb{R}) \\ gK & \mapsto & g \cdot {}^{t}g, \end{array}$$

where $G = \mathrm{GL}_{n+1}(\mathbb{R}),\ K = \mathrm{O}(n+1)$. This space G/K has the G-invariant Riemannian metric

$$ds^2 = \frac{1}{2}\,\mathrm{tr}\left((S^{-1}dS)^2\right).$$

Embedding $N_n \hookrightarrow \mathrm{Sym}^+_{n+1}(\mathbb{R})$

Put an affine subgroup

$$G_A := \left\{ \begin{pmatrix} P & \mu \\ 0 & 1 \end{pmatrix} \middle| P \in \mathrm{GL}_n(\mathbb{R}), \mu \in \mathbb{R}^n \right\} \subset \mathrm{GL}_{n+1}(\mathbb{R}).$$

Define a Riemannian submanifold as the orbit

$$G_A \cdot I_{n+1} = \{ g \cdot {}^t g |\ g \in G_A \} \subset \mathrm{Sym}^+_{n+1}(\mathbb{R}).$$

Theorem (Ref. [Calvo, Oller 2001])

We have the following isometry

$$\begin{aligned} N_n &\xrightarrow{\sim} G_A \cdot I_{n+1} \subset \mathrm{Sym}^+_{n+1}(\mathbb{R}), \\ (\Sigma, \mu) &\mapsto \begin{pmatrix} \Sigma + \mu {}^t\mu & \mu \\ {}^t\mu & 1 \end{pmatrix}. \end{aligned} \tag{3}$$

Embedding $N_n \hookrightarrow \mathrm{Sym}^+_{n+1}(\mathbb{R})$

By using the above embedding, we get a simpler expression of the metric and the geodesic equation.

	N_n	$\cong$	$G_A \cdot I_{n+1} \subset \mathrm{Sym}^+_{n+1}(\mathbb{R})$
coordinate	(Σ, μ)	$\mapsto$	$S = \begin{pmatrix} \Sigma + \mu {}^t\mu & \mu \\ {}^t\mu & 1 \end{pmatrix}$
metric	$ds^2 = ({}^t d\mu)\Sigma^{-1}(d\mu) + \frac{1}{2}\mathrm{tr}((\Sigma^{-1}d\Sigma)^2)$	$\Leftrightarrow$	$ds^2 = \frac{1}{2}\,\mathrm{tr}\left((S^{-1}dS)^2\right)$
geodesic eq.	$\begin{cases} \ddot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\mu} = 0, \\ \ddot{\Sigma} + \dot{\mu}{}^t\dot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\Sigma} = 0 \end{cases}$	$\Leftrightarrow$	$\underline{(I_n, 0)(S^{-1}\dot{S}) = (B, x)}$

Reconsideration of Eriksen's argument

We can interpret the Eriksen's argument as follows.

		Differential equation		Geodesic equation
		$\underline{\Lambda^{-1}\dot{\Lambda} = -A}$	$\longrightarrow$	$\underline{(I_n, 0)(S^{-1}\dot{S}) = (B, x)}$
$A = \begin{pmatrix} B & x & 0 \\ {}^t x & 0 & -{}^t x \\ 0 & -x & -B \end{pmatrix}$	$\longmapsto$	$e^{-tA} = \begin{pmatrix} \Delta & \delta & * \\ {}^t\delta & \epsilon & * \\ * & * & * \end{pmatrix}$	$\longmapsto$	$S := \begin{pmatrix} \Delta & \delta \\ {}^t\delta & \epsilon \end{pmatrix}^{-1}$
$\pitchfork$		$\pitchfork$		$\pitchfork$
$\{A : JAJ = -A\}$	$\longrightarrow$	$\{\Lambda : J\Lambda J = \Lambda^{-1}\}$	$\xrightarrow[\text{Essential!}]{}$	$N_n \cong G_A \cdot I_{n+1}$
$\cap$		$\cap$		$\cap$
$\mathrm{sym}_{2n+1}(\mathbb{R})$	$\xrightarrow[\exp]{}$	$\mathrm{Sym}^+_{2n+1}(\mathbb{R})$	$\xrightarrow[\text{projection}]{}$	$\mathrm{Sym}^+_{n+1}(\mathbb{R})$

Here $J = \begin{pmatrix} & & I_n \\ & 1 & \\ I_n & & \end{pmatrix}$.

Geodesic equation on Elliptical model

Definition

Let us define a Riemannian manifold $E_n(\alpha) = (M, ds^2)$ by

$$M = \left\{(\mu, \Sigma) \in \mathbb{R}^n \times \mathrm{Sym}_n^+(\mathbb{R})\right\},$$

$$ds^2 = ({}^t d\mu)\Sigma^{-1}(d\mu) + \frac{1}{2}\mathrm{tr}((\Sigma^{-1}d\Sigma)^2) + \frac{1}{2}d_\alpha\left(\mathrm{tr}(\Sigma^{-1}d\Sigma)\right)^2. \quad (4)$$

where $d_\alpha = (n+1)\alpha^2 + 2\alpha,\ \alpha \in \mathbb{C}$. Then $E_n(0) = N_n$.

The geodesic equation on $E_n(\alpha)$ is

$$\begin{cases} \ddot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\mu} = 0, \\ \ddot{\Sigma} + \dot{\mu}\,{}^t\dot{\mu} - \dot{\Sigma}\Sigma^{-1}\dot{\Sigma} - \dfrac{d_\alpha}{nd_\alpha + 1}\,{}^t\dot{\mu}\Sigma^{-1}\dot{\mu}\Sigma = 0. \end{cases} \quad (5)$$

This is equivalent to the geodesic equation on the elliptical model.

Geodesic equation on Elliptical model

The manifold $E_n(\alpha)$ is also embedded into positive-definite symmetric matrices $\mathrm{Sym}^+_{n+1}(\mathbb{R})$, ref. [Calvo, Oller 2001], and we have simpler expression of the geodesic equation.

	$E_n(\alpha)$	$\cong$	${}^{\exists}G_A(\alpha) \cdot I_{n+1} \subset \mathrm{Sym}^+_{n+1}(\mathbb{R})$
coordinate	(Σ, μ)	$\mapsto$	$S = \lvert\Sigma\rvert^{\alpha} \begin{pmatrix} \Sigma + \mu {}^t\mu & \mu \\ {}^t\mu & 1 \end{pmatrix}$
metric	(4)	$\Leftrightarrow$	$ds^2 = \frac{1}{2} \mathrm{tr}\left((S^{-1} dS)^2\right)$
geodesic eq.	(5)	$\Leftrightarrow$	$\underline{(I_n, 0)(S^{-1}\dot{S}) = (C, x) - \alpha(\log \lvert S\rvert)'(I_n, 0)}$

$|A| = \det A$

Geodesic equation on Elliptical model

But, in general, we do not ever construct any submanifold $N \subset \mathrm{Sym}^+_{2n+1}(\mathbb{R})$ such that its projection is $E_n(\alpha)$:

$$
\begin{array}{ccc}
\text{Differential equation} & & \text{Geodesic equation} \\
\underline{\Lambda^{-1}\dot{\Lambda} = -A} & \longrightarrow & \underline{(I_n, 0)(S^{-1}\dot{S}) = (C, x) - \alpha(\log|S|)'(I_n, 0)} \\
\Lambda(t) & \longmapsto & S(t) \\
\Cap & & \Cap \\
N & \longrightarrow & E_n(\alpha) \cong G_A(\alpha) \cdot I_{n+1} \\
\cap & & \cap \\
\mathrm{Sym}^+_{2n+1}(\mathbb{R}) & \underset{\text{projection}}{\longrightarrow} & \mathrm{Sym}^+_{n+1}(\mathbb{R})
\end{array}
$$

The geodesic equation on elliptical model has not been solved.

Future work

1. Extend Eriksen's theorem for elliptical models (ongoing)
2. Find Eriksen type theorem for general symmetric spaces G/K

   <u>**Sketch of the problem:**</u>

   For a projection $p : G/K \to G/K$,
   find a geodesic submanifold $N \subset G/K$,
   such that $p|_N$ maps all the geodesics to the geodesics:

$$\begin{array}{ccc} {}^{\forall}\Lambda(t)\text{: Geodesic} & \longmapsto & p(\Lambda(t))\text{: Geodesic} \\ \Cap & & \Cap \\ N & \xrightarrow[p|_N]{} & p(N) \\ \cap & & \cap \\ G/K & \xrightarrow[p\text{:projection}]{} & G/K \end{array}$$

References

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