Asymmetric Topologies on Statistical Manifolds

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Sources and Consequences of Asymmetry

Method: Symmetric Sandwich

Results

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Asymmetric Information Distances

Kullback-Leibler divergence

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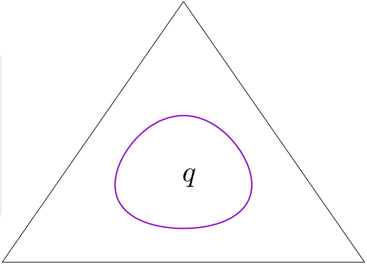

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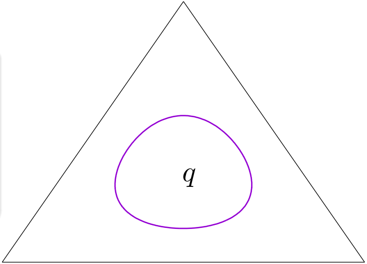

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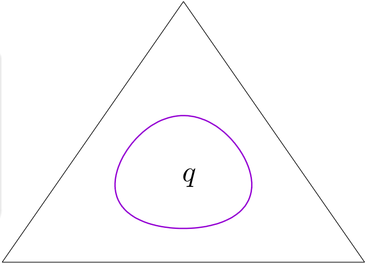

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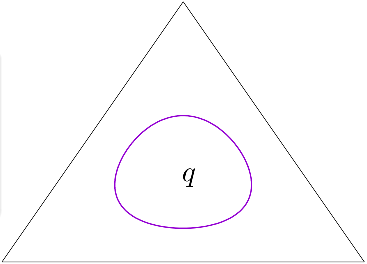

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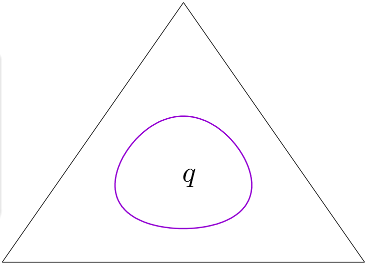

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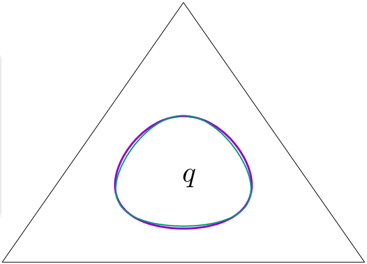

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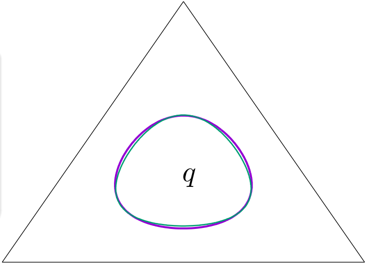

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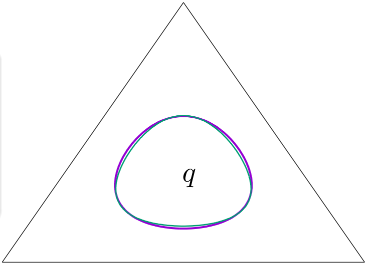

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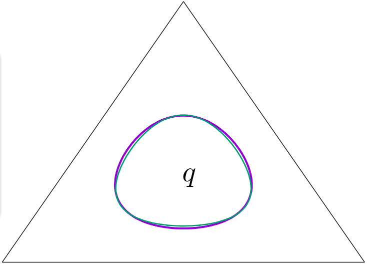

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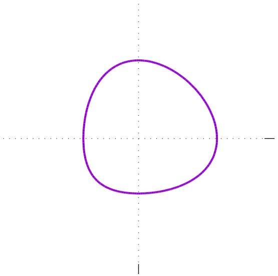
- Practically all other results have to be reconsidered (e.g. Baire category theorem, Alaoglu-Bourbaki, etc). (Cobzas, 2013).

Random Variables as the Source of Asymmetry

$M^\circ := \{x : \langle x, y\rangle \leq 1,\ \forall\, y \in M\}$

M



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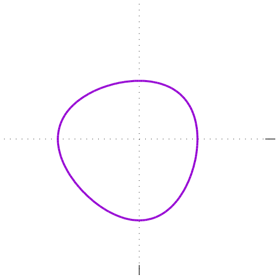

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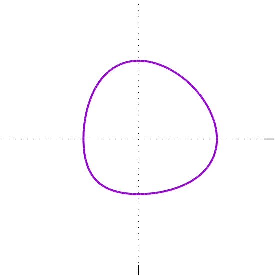

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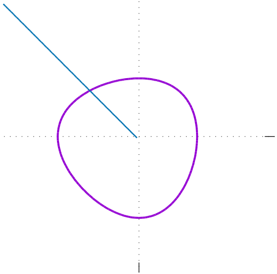

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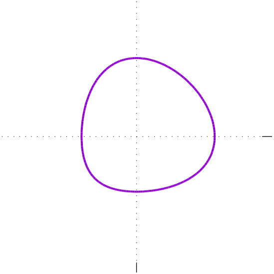

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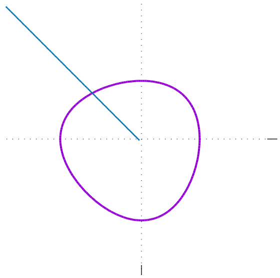

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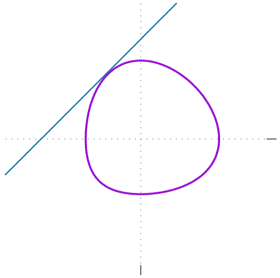

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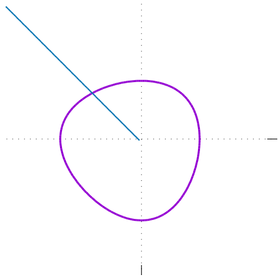

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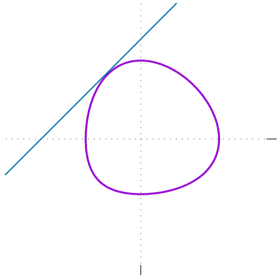

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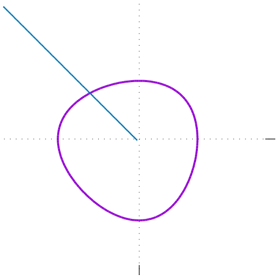

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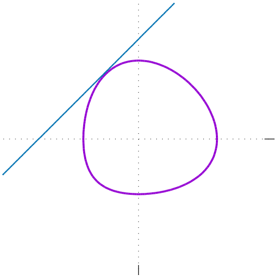

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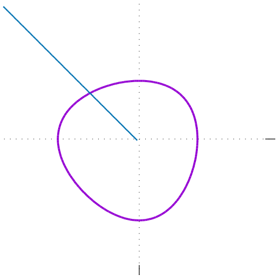

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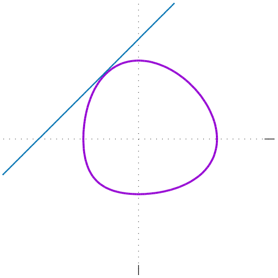

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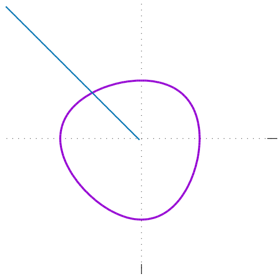

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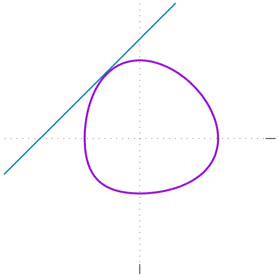

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- $0 \notin \operatorname{Int}(\operatorname{dom} \mathbb{E}_q\{e^x\})$

Example (Error minimization)

- Minimize $x = \frac{1}{2}\|a - b\|_2^2$ subject to $D_{KL}[w, q \otimes p] \leq \lambda$, $a, b \in \mathbb{R}^n$.
- $\mathbb{E}_w\{x\} < \infty$ minimized at $\overline{w} \propto e^{-\beta x} q \otimes p$.
- Maximization of x has no solution.
- $\frac{1}{2}\|a - b\|_2^2 \notin \operatorname{dom} \mathbb{E}_{q\otimes p}\{e^x\}$, $-\frac{1}{2}\|a - b\|_2^2 \in \operatorname{dom} \mathbb{E}_{q\otimes p}\{e^x\}$
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Sources and Consequences of Asymmetry

Method: Symmetric Sandwich

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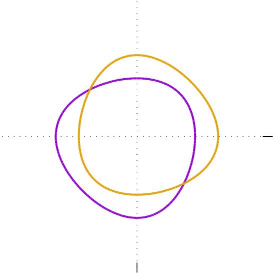

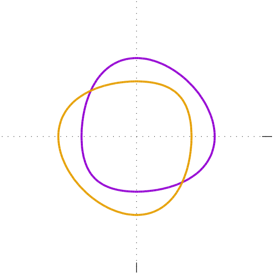

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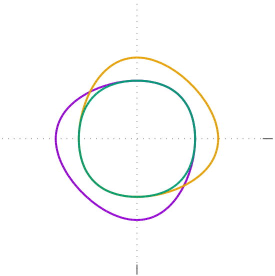
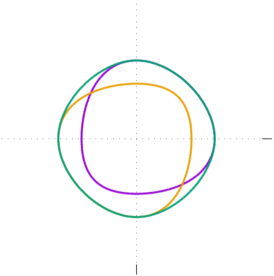


$\mu M^\circ \leq \mu(-M^\circ) \vee \mu M^\circ$



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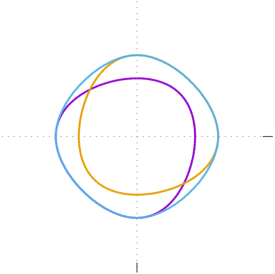

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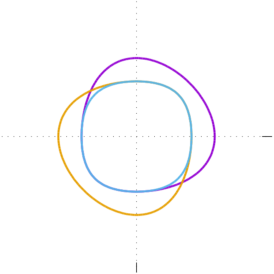

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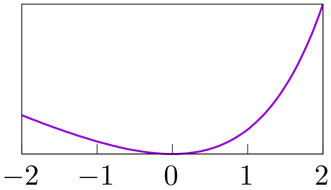
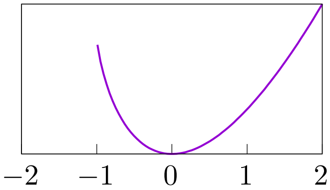
Lower and upper Luxemburg (Orlicz) norms





$$\varphi^*(x) = e^x - 1 - x \qquad \varphi(u) = (1+u)\ln(1+u) - u$$

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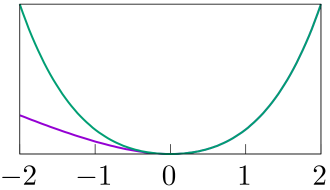

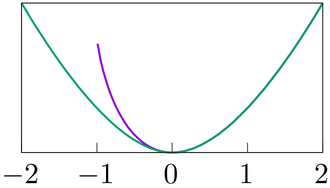

$$\begin{aligned}\varphi^*(x) &= e^x - 1 - x\\ \varphi^*_+(x) &= \varphi^*(|x|) \notin \Delta_2\end{aligned}$$

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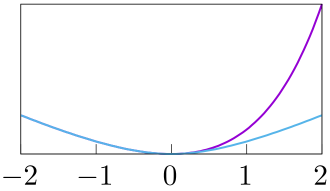

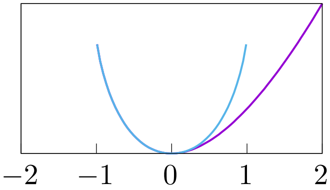

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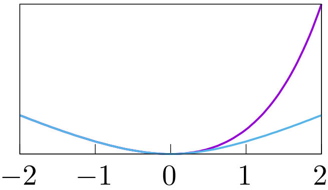

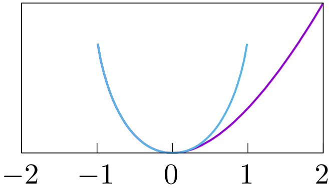

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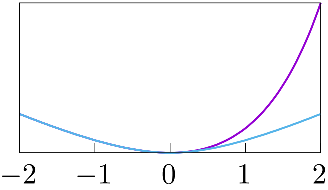

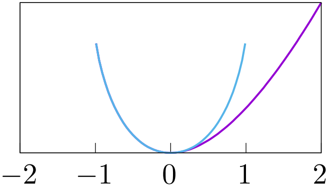

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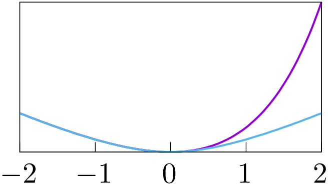

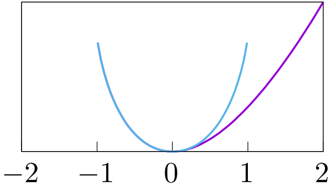

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Sources and Consequences of Asymmetry

Method: Symmetric Sandwich

Results

KL Induces Hausdorff (T_2) Asymmetric Topology

Theorem

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Proof.

$\|u\|_{\varphi+} \leq \|u|_\varphi$ (resp. $\|x\|_{\varphi-} \leq \|x|_\varphi$) implies $(Y, \| \cdot |_\varphi)$ (resp. $(X, \| \cdot |^*_\varphi)$) is finer than normed space $(Y, \| \cdot \|_{\varphi+})$ (resp. $(X, \| \cdot \|^*_{\varphi-})$). □

Separable Subspaces

Theorem

$(Y, \|\cdot\|_{\varphi+})$ *(resp.* $(X, \|\cdot\|^*_{\varphi-})$*) is a separable Orlicz subspace of* $(Y, \|\cdot|_{\varphi})$ *(resp.* $(X, \|\cdot|^*_{\varphi})$*).*

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$\varphi_+(u) = (1 + |u|)\ln(1 + |u|) - |u| \in \Delta_2$ (resp. $\varphi^*_-(x) = e^{-|x|} - 1 + |x| \in \Delta_2$). Note that $\varphi_- \notin \Delta_2$ and $\varphi^*_+ \notin \Delta_2$. □

Completeness

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$(Y, \|\cdot|_\varphi)$ *(resp.* $(X, \|\cdot|^*_\varphi)$*) is*

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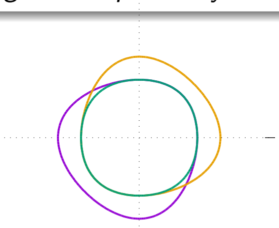

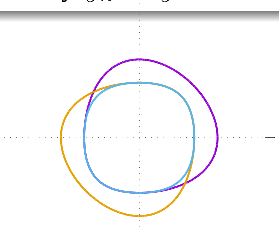

Proof.

$\rho^s(y,z) = \|z-y|_\varphi \vee \|y-z|_\varphi \le \|y-z\|_{\varphi-}$, where $(Y, \|\cdot\|_{\varphi-})$ is Banach.
Then use theorems of Reilly et al. (1982) and Chen et al. (2007). □

Summary and Further Questions

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- Other asymmetric information distances (e.g. Renyi divergence).

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Method: Symmetric Sandwich

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