SYMMETRY METHODS IN GEOMETRIC MECHANICS

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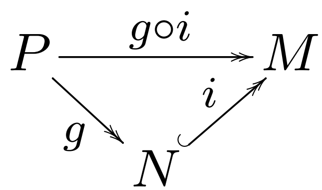
PLAN OF THE PRESENTATION

- Lie group actions and reduction of dynamics
- The above in the Hamiltonian case
- Properties of the momentum map
- Regular reduction
- Singular reduction
- Regular cotangent bundle reduction

M, N manifolds, $N \subset M$ as subsets.

N is an **initial submanifold** of M if the inclusion map $i : N \hookrightarrow M$ is an immersion satisfying the following condition: for any smooth manifold P and any map $g : P \to N$, g is smooth if and only if $i \circ g : P \to M$ is smooth. The smooth manifold structure that makes N into an initial submanifold of M is unique.



The integral manifolds of an integrable generalized distribution (thus forming a generalized foliation) are initial.

Infinitesimal generator $\xi_M \in \mathfrak{X}(M)$ associated to $\xi \in \mathfrak{g}$: Lie(G)

$$\xi_M(m) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp t\xi}(m) = T_e\Phi^m \cdot \xi.$$

ξ_M is a complete vector field with flow $(t, m) \mapsto \exp t\xi \cdot m$.
$\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a **Lie algebra antihomomorphism**

Isotropy, **stabilizer**, **symmetry subgroup** of $m \in M$

$$G_m := \{g \in G \mid \Phi_g(m) = m\} \subset G, \qquad G_{g\cdot m} = gG_m g^{-1},\ \forall g \in G$$

closed subgroup of G whose Lie algebra $\mathfrak{g}_m$ equals

$$\mathfrak{g}_m = \{\xi \in \mathfrak{g} \,|\, \xi_M(m) = 0\}.$$

$\mathcal{O}_m \equiv G \cdot m := \{\Phi_g(m) \,|\, g \in G\}$ **G-orbit** of m

$\mathcal{O}_m \ni g \cdot m \overset{\sim}{\longleftrightarrow} gG_m \in G/G_m$ diffeomorphism

$\mathcal{O}_m$ initial submanifold of M

- **Transitive action**: only one orbit, that is, $\mathcal{O}_m = M$
- **Free action**: $G_m = \{e\}$ for all $m \in M$
- **Proper action**: if $\overline{\Phi} : G \times M \ni (g, m) \longmapsto (m, g \cdot m) \in M \times M$ is proper. Equivalent to: for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n\}$ in M, there exists a convergent subsequence $\{g_{n_k}\}$ in G.

Examples of proper actions: compact group actions, $SE(n)$ acting on $\mathbb{R}^n$, Lie groups acting on themselves by translation.

Fundamental facts about proper Lie group actions

(i) The isotropy subgroups G_m are compact.
(ii) The orbit space M/G is a Hausdorff topological space.
(iii) If the action is free, M/G is a smooth manifold, and the canonical projection $\pi : M \to M/G$ defines on M the structure of a smooth left principal G–bundle.
(iv) If all the isotropy subgroups of the elements of M under the G–action are conjugate to a given subgroup H, then M/G is a smooth manifold and $\pi : M \to M/G$ defines the structure of a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber G/H.
Normalizer of H is $N(H) := \{g \in G \mid gH = Hg\}$.
(v) If the manifold M is paracompact then there exists a G-invariant Riemannian metric on it. (Palais)
(vi) If the manifold M is paracompact then smooth G-invariant functions separate the G-orbits.

Twisted product

$H \subset G$ Lie subgroup acting (left) on the manifold A. **Right twisted action** of H on $G \times A$, defined by

$$(g, a) \cdot h = (gh, h^{-1} \cdot a), \quad g, h \in G, \quad a \in A,$$

is free and proper. **Twisted product** $G \times_H A := (G \times A)/H$.

Tube

G acts properly on M. For $m \in M$, let $H := G_m$. A **tube** around the orbit $G \cdot m$ is a G-equivariant diffeomorphism

$$\varphi : G \times_H A \longrightarrow U,$$

where U is a G-invariant neighborhood of $G \cdot m$ and A is some manifold on which H acts.

Slice Theorem

Let G be a Lie group acting properly on M at the point $m \in M$, $H := G_m$. Then there exists a tube

$$\varphi : G \times_H B \longrightarrow U$$

about $G \cdot m$. B is an open H-invariant neighborhood of 0 in a vector space which is H-equivariantly isomorphic to $T_m M / T_m(G \cdot m)$, where the H-representation is given by

$$h \cdot (v + T_m(G \cdot m)) := T_m \Phi_h \cdot v + T_m(G \cdot m).$$

Slice

$S := \varphi([e, B])$ so that $U = G \cdot S$.

From now on, we assume that G acts on M properly.

Dynamical consequences

Let $X \in \mathfrak{X}(U)^G$, $U \subset M$ open G-invariant, S slice at $m \in U$. Then:

- $\exists\, X_T \in \mathfrak{X}(G\cdot S)^G$, $X_T(z) = \xi(z)_M(z)$ for $z \in G\cdot S$, where $\xi : G\cdot S \to \mathfrak{g}$ is smooth G-equivariant and $\xi(z) \in \mathrm{Lie}(N(G_z))$ for all $z \in G\cdot S$. The flow T_t of X_T is given by $T_t(z) = \exp t\xi(z)\cdot z$, so X_T is complete.
- $\exists\, X_N \in \mathfrak{X}(S)^{G_m}$.
- If $z = g\cdot s$, for $g \in G$ and $s \in S$, then

$$X(z) = X_T(z) + T_s\Phi_g\left(X_N(s)\right) = T_s\Phi_g\left(X_T(s) + X_N(s)\right)$$

- If N_t is the flow of X_N (on S) then the integral curve of $X \in \mathfrak{X}(U)^G$ through $g\cdot s \in G\cdot S$ is

$$F_t(g\cdot s) = g(t)\cdot N_t(s),$$

where $g(t) \in G$ is the solution of

$$\dot{g}(t) = T_eL_{g(t)}\Big(\xi(N_t(s))\Big), \qquad g(0) = g.$$

This is the **tangential-normal** decomposition of a G-invariant vector field (or **Krupa decomposition** in bifurcation theory).

Geometric consequences

$$\begin{aligned} M_{(H)} &= \{z \in M \mid G_z \in (H)\}, && \textbf{orbit type set} \\ M^H &= \{z \in M \mid H \subset G_z\}, && \textbf{fixed point set} \\ M_H &= \{z \in M \mid H = G_z\}, && \textbf{isotropy type set} \end{aligned}$$

are (embedded) submanifolds of M, M_H open in M^H, but, in general, M_H is not closed in M.

Let $N(H) := \{g \in G \mid gH = Hg\}$ be the normalizer of H in G. $N(H)/H$ acts freely and properly on M_H.

$m \in M$ is **regular** if $\exists U \ni m$ such that $\dim \mathcal{O}_z = \dim \mathcal{O}_m, \forall z \in U$.

Principal Orbit Theorem: M connected. $M^{reg} := \{m \in M \mid m \text{ regular}\}$ is connected, open, and dense in M. M/G contains only one principal orbit type, which is connected, open, dense in M/G.

The Stratification Theorem: The connected components of all orbit type manifolds $M_{(H)}$ and their projections onto $M_{(H)}/G$ constitute a Whitney stratification of M and M/G, respectively. This stratification of M/G is minimal among all Whitney stratifications of M/G.

G-Codostribution Theorem: Let G be a Lie group acting properly on the smooth manifold M and $m \in M$ a point with isotropy subgroup $H := G_m$. Then

$$\left(\left(T_m(G \cdot m)\right)^\circ\right)^H = \left\{\mathbf{d}f(m) \mid f \in C^\infty(M)^G\right\}.$$

This is due to Ortega [1998].

Reduction of general vector fields

$G \times M \to M$ proper, $X \in \mathfrak{X}(M)^G$ (G-equivariant) with flow F_t

Law of conservation of isotropy: Every isotropy type submanifold $M_H := \{m \in M \mid G_m = H\}$ is preserved by F_t.

$\pi_H : M_H \to M_H/(N(H)/H)$ projection , $i_H : M_H \hookrightarrow M$ inclusion

X induces a unique H-**isotropy type reduced vector field** X^H on $M_H/(N(H)/H)$ by

$$X^H \circ \pi_H = T\pi_H \circ X \circ i_H,$$

whose flow F_t^H is given by

$$F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H.$$

G compact linear action, then the construction of $M_H/(N(H)/H)$ can be implemented by using the invariant polynomials of the action and the theorems of Hilbert and Schwarz-Mather.

The Hamiltonian case

(M,ω) symplectic manifold, G connected Lie group with Lie algebra $\mathfrak{g}$, $G \times M \to M$ left free proper symplectic action: $\Phi_g^*\omega = \omega$, $\forall g \in G$.

$\mathbf{J} : M \to \mathfrak{g}^*$ **momentum map**: $X_{\mathbf{J}^\xi} = \xi_M$, where $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$.
Non-equivariance (Souriau) group $\mathfrak{g}^*$-valued 1-cocycle:

$c(g) := \mathbf{J}(g \cdot m) - \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(m)$, independent of $m \in M$ if M connected.

(M,ω) connected. $G \times \mathfrak{g}^* \ni (g,\mu) \stackrel{\Theta}{\longmapsto} \mathrm{Ad}^*_{g^{-1}} \mu + c(g) \in \mathfrak{g}^*$ **affine action**. $\mathbf{J} : M \to \mathfrak{g}^*$ is Θ-equivariant.

Noether's Theorem: $\mathbf{J}$ is conserved along the flow of any G-invariant Hamiltonian.

$\mathfrak{g}^*_\pm$ is an affine Lie-Poisson space

$$\{f,h\}(\mu) := \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle \mp \Sigma \left(\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right), \quad f, h \in C^\infty(\mathfrak{g}^*)$$

The **infinitesimal non-equivariance two-cocycle** $\Sigma \in Z^2(\mathfrak{g}, \mathbb{R})$ is

$$\Sigma : \mathfrak{g} \times \mathfrak{g} \ni (\xi, \eta) \longmapsto \mathbf{d}\widehat{\sigma}_\eta(e) \cdot \xi \in \mathbb{R},$$

where $\widehat{\sigma}_\eta : G \to \mathbb{R}$ defined by $\widehat{\sigma}_\eta(g) = \langle \sigma(g), \eta \rangle$.

Its symplectic leaves (reachable sets) are the Θ-orbits $\mathcal{O}_\mu$:

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \pm\langle \nu, [\xi, \eta] \rangle \mp \Sigma(\xi, \eta).$$

$\mathbf{J} : M \to \mathfrak{g}^*_+$ is a Poisson map.

Example: lifted actions on cotangent bundles. G acts on the manifold Q and then by lift on its cotangent bundle T^*Q.

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle,$$

$\forall\, \alpha_q \in T^*Q$, $\forall\, \xi \in \mathfrak{g}$. This is an Ad^*-equivariant momentum map.

Special case 1: linear momentum. Configuration space of N particles in space is $\mathbb{R}^{3N}$. $\mathbb{R}^3$ acts on $\mathbb{R}^{3N}$ by $\mathbf{v} \cdot (\mathbf{q}_i,) = (\mathbf{q}_i + \mathbf{v})$. Then $\mathbf{J} : T^*\mathbb{R}^{3N} \to \mathbb{R}^3$ is the linear momentum $\mathbf{J}(\mathbf{q}_i, \mathbf{p}^i) = \sum_{i=1}^N \mathbf{p}^i$.

Special case 2: angular momentum. SO(3) acts naturally on $\mathbb{R}^3$. Then $\mathbf{J} : T^*\mathbb{R}^{3N} \to \mathbb{R}^3$ is the angular momentum $\mathbf{J}(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p}$.

Example: symplectic linear actions. (V, ω) symplectic vector space, $G \subseteq \mathrm{Sp}(V, \omega)$, acting naturally on V. Ad^*-equivariant momentum map $\mathbf{J} : V \to \mathfrak{sp}(V, \omega)^*$ is

$$\langle \mathbf{J}(v),\, \xi \rangle = \frac{1}{2}\omega(\xi_V(v),\, v).$$

Special case: Cayley-Klein parameters and the Hopf fibration. SU(2) acts on $\mathbb{C}^2$, $\mathbf{J} : \mathbb{C}^2 \to \mathfrak{su}(2)^*$ given, as above, by

$$\langle \mathbf{J}(z, w), \xi \rangle = \frac{1}{2}\omega(\xi(z, w)^{\mathsf{T}}, (z, w)), \quad z, w \in \mathbb{C}, \;\; \xi \in \mathfrak{su}(2).$$

Lie algebra isomorphism $(\mathfrak{su}(2), [\,,])\rightarrow(\mathbb{R}^3, \times)$ given by

$$\mathbb{R}^3 \ni \mathbf{x} = (x^1, x^2, x^3) \stackrel{\sim}{\longleftrightarrow}$$

$$\tilde{\mathrm{x}} := \frac{1}{2}\begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix} \in \mathfrak{su}(2).$$

Identify $\mathfrak{su}(2)^*$ with $\mathbb{R}^3$ by the map $\mu \in \mathfrak{su}(2)^* \mapsto \breve{\mu} \in \mathbb{R}^3$ defined by

$$\breve{\mu}\cdot\mathbf{x} := -2\langle\mu, \tilde{\mathbf{x}}\rangle, \quad \forall\, \mathbf{x} \in \mathbb{R}^3.$$

Then $\breve{\mathbf{J}} : \mathbb{C}^2 \rightarrow \mathbb{R}^3$ has the expression

$$\breve{\mathbf{J}}(z, w) = -\frac{1}{2}(2w\bar{z}, |z|^2 - |w|^2) \in \mathbb{R}^3.$$

(z, w) are the **Cayley-Klein parameters** or the **Kustaanheimo-Stiefel coordinates**. $\breve{\mathbf{J}}|_{S^3} : S^3 \rightarrow S^2_{1/2}$ is the Hopf fibration. Similar construction in fluid dynamics: **Clebsch variables**.

The momentum map of the SU(2)-action on $\mathbb{C}^2$, the Cayley-Klein parameters, the Kustaanheimo-Stiefel coordinates, and the family of Hopf fibrations on concentric three-spheres in $\mathbb{C}^2$ are ***the same map****.*

Properties of the momentum map

• range $T_m\mathbf{J} = (\mathfrak{g}_m)^\circ$. Points with symmetry are points of bifurcation. Freeness of the action is equivalent to the regularity of $\mathbf{J}$.

• $\ker T_m\mathbf{J} = (\mathfrak{g} \cdot m)^\omega$.

• The obstruction to the existence of $\mathbf{J}$ is the vanishing of the map $H^1(\mathfrak{g}, \mathbb{R}) := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \ni [\xi] \longmapsto \left[\mathbf{i}_{\xi_M}\omega\right] \in H^1(M, \mathbb{R})$.

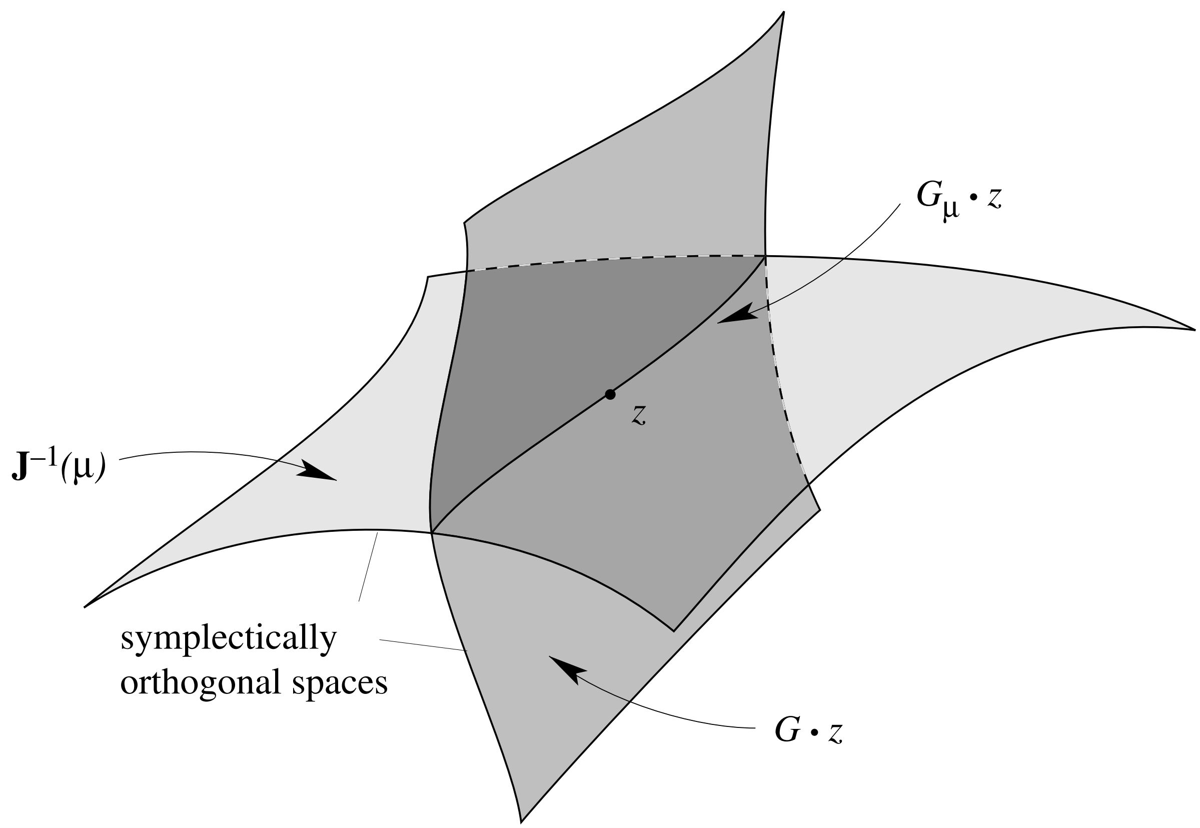
• $\mathbf{J}^{[\xi,\eta]} = \{\mathbf{J}^\xi, \mathbf{J}^\eta\} \iff T_m\mathbf{J}\left(\xi_M(m)\right) = -\operatorname{ad}^*_\xi \mathbf{J}(m)\ \forall m \in M, \xi, \eta \in \mathfrak{g}$
Among all possible choices of momentum maps for a given action, there is at most one infinitesimally Ad*-equivariant one.

G connected, then infinitesimal Ad*-equivariance $\Leftrightarrow$ Ad*-equivariance.

- $H^1(\mathfrak{g};\mathbb{R}) = 0$ or $H^1(M,\mathbb{R}) = 0 \Rightarrow \mathbf{J}$ exists. $H^2(\mathfrak{g};\mathbb{R}) = 0 \Rightarrow \mathbf{J}$ equiv.
Whitehead lemmas: $\mathfrak{g}$ is semisimple $\Longrightarrow H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$.

- If G is compact $\mathbf{J}$ can always be chosen to be Ad^*-equivariant

- **Reduction Lemma:** $\mathfrak{g}_{\mathbf{J}(m)} \cdot m = \mathfrak{g} \cdot m \cap \ker T_m\mathbf{J} = \mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega}$.



Momentum maps and isotropy type manifolds

- M_{G_m} is a symplectic submanifold of M for any $m \in M$.

This is based on: H compact Lie group and (V, ω) symplectic representation space. Then V^H is a symplectic subspace of V.

- Let $M^m_{G_m}$ be the connected component of M_{G_m} containing m and

$$N(G_m)^m := \{n \in N(G_m) \mid n \cdot z \in M^m_{G_m} \text{ for all } z \in M^m_{G_m}\}.$$

$N(G_m)^m$ is a closed subgroup of $N(G_m)$ that contains the connected component of the identity. So it is also open and hence $\text{Lie}(N(G_m)^m) = \text{Lie}(N(G_m))$.

In addition, $(N(G_m)/G_m)^m = N(G_m)^m/G_m$ so that

$$\text{Lie}\left(N(G_m)^m/G_m\right) = \text{Lie}\left(N(G_m)/G_m\right).$$

- $L^m := N(G_m)^m/G_m$ acts freely properly and canonically on $M^m_{G_m}$ by $\Psi(nG_m, z) := n \cdot z$.

• The free proper canonical action of $L^m := N(G_m)^m/G_m$ on $M^m_{G_m}$ has a momentum map $\mathbf{J}_{L^m} : M^m_{G_m} \to (\mathrm{Lie}(L^m))^*$ given by

$$\mathbf{J}_{L^m}(z) := \Lambda(\mathbf{J}|_{M^m_{G_m}}(z) - \mathbf{J}(m)), \quad z \in M^m_{G_m}.$$

In this expression $\Lambda : (\mathfrak{g}^\circ_m)^{G_m} \to (\mathrm{Lie}(L^m))^*$ denotes the natural L^m-equivariant isomorphism given by

$$\left\langle \Lambda(\beta), \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)\, G_m \right\rangle = \langle \beta, \xi \rangle,$$

for any $\beta \in (\mathfrak{g}^\circ_m)^{G_m}$, $\xi \in \mathrm{Lie}(N(G_m)^m) = \mathrm{Lie}(N(G_m))$.

• The non-equivariance one-cocycle $\tau : M^m_{G_m} \to (\mathrm{Lie}(L^m))^*$ of the momentum map $\mathbf{J}_{L^m}$ is given by the map

$$\tau(l) = \Lambda(c(n) + n \cdot \mathbf{J}(m) - \mathbf{J}(m)), \quad l = nG_m \in L^m,\ n \in N(G_m)^m.$$

So, even if $\mathbf{J}$ is equivariant, the induced momentum map $\mathbf{J}_{L_m}$ is not, in general!

Convexity

$\mathbf{J} : M \to \mathfrak{g}^*$ equivariant, G, M compact connected. The intersection of range $\mathbf{J}$ with a Weyl chamber is a *compact and convex polytope*, the **momentum polytope** (Atiyah, Guillemin, Kirwan, Sternberg).

Delzant polytope in $\mathbb{R}^n$ is a convex polytope that is also:
(i) Simple: there are n edges meeting at each vertex.
(ii) Rational: the edges meeting at a vertex p are of the form $p + tu_i$, $0 \leq t < \infty$, $u_i \in \mathbb{Z}^n$, $i \in \{1, \ldots, n\}$.
(iii) Smooth: the vectors $\{u_1, \ldots, u_n\}$ can be chosen to be an integral basis of $\mathbb{Z}^n$.

Delzant's Theorem: There is a biection

$$\{\text{symplectic toric manifolds}\} \stackrel{\sim}{\longleftrightarrow} \{\text{Delzant polytopes}\}$$
$$(M, \omega, \mathbb{T}^n, \mathbf{J} : M \to \mathbb{R}^n) \stackrel{\sim}{\longleftrightarrow} \mathbf{J}(M)$$

Marsden-Weinstein Reduction Theorem

• If $\mu \in \mathbf{J}(M) \subset \mathfrak{g}^*$ regular value of $\mathbf{J}$ and
• G_μ-action on $\mathbf{J}^{-1}(\mu)$ is free and proper; $G_\mu := \{g \in G \mid \Theta_g \mu = \mu\}$,
then $(M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu, \omega_\mu)$ is symplectic:

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega,$$

$i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$ inclusion $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ projection.

The flow F_t of X_h, $h \in C^\infty(M)^G$, leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the G-action, so it induces a flow F_t^μ on M_μ by

$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu.$$

F_t^μ is Hamiltonian on (M_μ, ω_μ) for the **reduced Hamiltonian** $h_\mu \in C^\infty(M_\mu)$ given by

$$h_\mu \circ \pi_\mu = h \circ i_\mu.$$

Moreover, if $h, k \in C^\infty(M)^G$, then $\{h, k\}_\mu = \{h_\mu, k_\mu\}_{M_\mu}$.

Orbit symplectic form from reduction

G a Lie group, $L_g(h) = gh$, $R_g(h) = hg$, left and right translations

$\mathcal{O}_\mu := \{\mathrm{Ad}^*_g \mu \mid g \in G\}$ coadjoint G-orbit through $\mu \in \mathfrak{g}^*$

Take the special case $M = G$ and the left action $g \cdot h := gh$, for all $g, h \in G$. The momentum map $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$ has the expression

$$\mathbf{J}_L(\alpha_g) = T_e^* R_g(\alpha_g) \in \mathfrak{g}^*, \quad \forall \alpha_g \in T^*G.$$

Then, $(\mathbf{J}_L^{-1}(\mu)/G_\mu, \Omega_\mu) \cong (\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$; **orbit symplectic form** is

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\mathrm{ad}^*_\xi \nu, \mathrm{ad}^*_\eta \nu) = \pm\langle \nu, [\xi, \eta]\rangle, \quad \forall\, \xi, \eta \in \mathfrak{g}, \ \nu \in \mathcal{O}_\mu$$

$\mathfrak{g}^*$ is a Lie-Poisson space for the bracket $((T^*G)/G \xleftrightarrow{\widehat{\mathbf{J}_R}} \mathfrak{g}^*_-)$

$$\{f, h\}(\mu) := \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right]\right\rangle, \quad f, h \in C^\infty(\mathfrak{g}^*)$$

and its symplectic leaves (reachable sets) are $\mathcal{O}_\mu$.

Reconstruction of dynamics

Given is an integral curve $c_\mu(t)$ of $X_{h_\mu} \in \mathfrak{X}(M_\mu)$. Let $m_0 \in \mathbf{J}^{-1}(\mu)$. Find integral curve $c(t)$ of $X_h \in \mathfrak{X}(M)$ with initial condition m_0.

Pick a smooth curve $d(t) \subset \mathbf{J}^{-1}(\mu)$ such that $d(0) = m_0$ and $\pi_\mu(d(t)) = c_\mu(t)$. If $c(t)$ is the integral curve of X_h with initial condition $c(0) = m_0$, then there is a curve $g(t) \subset G_\mu$ such that $c(t) = g(t) \cdot d(t)$.

1.) Find smooth curve $\xi(t) \subset \mathfrak{g}_\mu$ s.t. $\xi(t)_M(d(t)) = X_h(d(t)) - \dot{d}(t)$.

2.) With this $\xi(t)$, solve $\dot{g}(t) = T_e L_{g(t)} \xi(t)$, $g(0) = e$.

Let $\mathcal{A} \in \Omega^1\left(\mathbf{J}^{-1}(\mu); \mathfrak{g}_\mu\right)$ be a connection on the G_μ-principal bundle $\mathbf{J}^{-1}(\mu) \to M_\mu$. Choose $d(t)$ to be the horizontal lift of $c_\mu(t)$ through m_0, i.e., $\mathcal{A}(d(t))(\dot{d}(t)) = 0$, $\pi_\mu(d(t)) = c_\mu(t)$, $d(0) = m_0$.

Then the solution of **1.)** is

$$\xi(t) = \mathcal{A}(d(t))\Big(X_h(d(t))\Big).$$

Orbit reduction

- $\Phi : G \times M \to M$ free proper symplectic action.
- The action admits a momentum map $\mathbf{J} : M \to \mathfrak{g}^*$.
- M is connected; if $\mathbf{J}$ equivariant, this is not needed.
- The affine coadjoint orbit

$$\mathcal{O}_\mu := \{\mathrm{Ad}^*_{g^{-1}}\mu + c(g) \mid g \in G\}$$

is an initial submanifold of $\mathfrak{g}^*$.

- Bifurcation Lemma (range $(T_m\mathbf{J}) = (\mathfrak{g}_m)^\circ$) + the freeness of the action (hence $\mathfrak{g}_m = \{0\}$) $\Longrightarrow$ $\mathbf{J}$ is a submersion onto some open subset of $\mathfrak{g}^*$. So $\mathbf{J}$ is transversal to $\mathcal{O}_\mu$, i.e., for any $z \in \mathbf{J}^{-1}(\mathcal{O}_\mu)$, we have $(T_z\mathbf{J})(T_zM) + T_{\mathbf{J}(z)}\mathcal{O}_\mu = \mathfrak{g}^*$. So $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is an initial submanifold of M of dimension

$$\dim(\mathbf{J}^{-1}(\mathcal{O}_\mu)) = \dim M - \dim G_\mu$$

whose tangent space at $z \in \mathbf{J}^{-1}(\mathcal{O}_\mu)$ equals

$$T_z(\mathbf{J}^{-1}(\mathcal{O}_\mu)) = (T_z\mathbf{J})^{-1}(T_{\mathbf{J}(z)}\mathcal{O}_\mu) = \mathfrak{g} \cdot z + \ker(T_z\mathbf{J}).$$

• G-action restricts to a free and proper G-action on the G-invariant initial submanifold $\mathbf{J}^{-1}(\mathcal{O}_\mu)$. Why is this restricted action smooth?

Action $\Phi : G \times M \to M$ is smooth, so $\Phi^{\mathcal{O}_\mu} : G \times \mathbf{J}^{-1}(\mathcal{O}_\mu) \to M$ is smooth (restriction: composition of smooth maps, $\mathbf{J}^{-1}(\mathcal{O}_\mu) \to M$). But $\Phi^{\mathcal{O}_\mu}(G \times \mathbf{J}^{-1}(\mathcal{O}_\mu)) \subset \mathbf{J}^{-1}(\mathcal{O}_\mu)$. Since $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is initial, it follows that $\Phi^{\mathcal{O}_\mu} : G \times \mathbf{J}^{-1}(\mathcal{O}_\mu) \to \mathbf{J}^{-1}(\mathcal{O}_\mu)$ is smooth.

• Hence $M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is a manifold and the projection $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}$ is a surjective submersion.

(i) On $M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ there is a unique symplectic form $\omega_{\mathcal{O}_\mu}$ characterized by $\iota^*_{\mathcal{O}_\mu}\omega = \pi^*_{\mathcal{O}_\mu}\omega_{\mathcal{O}_\mu} + \mathbf{J}^*_{\mathcal{O}_\mu}\omega^+_{\mathcal{O}_\mu}$.

$\iota_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \hookrightarrow M$, $\mathbf{J}_{\mathcal{O}_\mu} := \mathbf{J}|_{\mathbf{J}^{-1}(\mathcal{O}_\mu)}$, and $\omega^+_{\mathcal{O}_\mu}$ is the $+$-symplectic structure on the affine orbit $\mathcal{O}_\mu$.

$(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is the **symplectic orbit reduced space**.

(ii) $h \in C^\infty(M)^G$. The flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ invariant and commutes with the G-action, so it induces a flow $F_t^{\mathcal{O}_\mu}$ on $M_{\mathcal{O}_\mu}$, uniquely determined by

$$\pi_{\mathcal{O}_\mu} \circ F_t \circ i_{\mathcal{O}_\mu} = F_t^{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu}.$$

(iii) The vector field generated by the flow $F_t^{\mathcal{O}_\mu}$ on $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is Hamiltonian with associated **reduced Hamiltonian** $h_{\mathcal{O}_\mu} \in C^\infty(M_{\mathcal{O}_\mu})$ defined by $h_{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu} = h \circ i_{\mathcal{O}_\mu}$. The vector fields X_h and $X_{h_{\mathcal{O}_\mu}}$ are $\pi_{\mathcal{O}_\mu}$-related.

(iv) $h, k \in C^\infty(M)^G \Rightarrow \{h, k\} \in C^\infty(M)^G$ and $\{h, k\}_{\mathcal{O}_\mu} = \{h_{\mathcal{O}_\mu}, k_{\mathcal{O}_\mu}\}_{M_{\mathcal{O}_\mu}}$, where $\{\cdot, \cdot\}_{M_{\mathcal{O}_\mu}}$ denotes the Poisson bracket associated to the symplectic form $\omega_{\mathcal{O}_\mu}$ on $M_{\mathcal{O}_\mu}$.

This is a theorem in the Poisson category whereas the point reduction theorem is in the symplectic category.

Problems with the hypotheses of the Reduction Theorem

The hypotheses are too restrictive, even in classical examples, such as Jacobi's elimination of the nodes. Properness of the action cannot be eliminated because one needs the theory of G-manifolds.

1.) How does one recover the conservation of isotropy? The momentum map seems incapable to get this. $\mathbf{J}^{-1}(\mu)$ are not the smallest invariant sets. Reduction completely ignores this point.

2.) If the G-action is not free, M_μ is not a smooth manifold. Then what is the structure of the reduced topological space? What is left that remains symplectic?

3.) If G is discrete, the momentum map is zero. What is reduction in that case?

These are questions in bifurcation theory with symmetry. For generic vector fields, a lot is known. For Hamiltonian vector fields, almost nothing (a few papers).

Singular point reduction

Given: (M, ω) connected, $m \in M$
G acting symplectically on M
$\mathbf{J} : (M, \omega) \to \mathfrak{g}^*$ momentum map
$c : G \to \mathfrak{g}^*$ group 1-cocycle defined by $c(g) := \mathbf{J}(g \cdot z) - \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(z)$
affine G-action $\Theta(g, \nu) := \mathrm{Ad}^*_{g^{-1}} \nu + c(g)$ on $\mathfrak{g}^*$
G_μ the Θ-isotropy at μ

Notation: M_H^m connected component of M_H containing m,

$H := G_m \subseteq G$

$\mu := \mathbf{J}(m) \in \mathfrak{g}^*$

Singular symplectic point strata

(i) $\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)$ is embedded in M.

(ii) $M_\mu^{(H)} := [\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)]/G_\mu$ has a unique quotient manifold structure such that

$$\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \longrightarrow M_\mu^{(H)}$$

is a surjective submersion.

(iii) There is a unique symplectic form $\omega_\mu^{(H)}$ on $M_\mu^{(H)}$ characterized by

$$\iota_\mu^{(H)\,*}\omega = \pi_\mu^{(H)\,*}\omega_\mu^{(H)},$$

$\iota_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \hookrightarrow M$ inclusion. $(M_\mu^{(H)}, \omega_\mu^{(H)})$ are the **singular symplectic point strata**.

(iv) $h \in C^\infty(M)^G$. Flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)$ invariant and commutes with the G_μ-action, so it induces flow F_t^μ on $M_\mu^{(H)}$:

$$\pi_\mu^{(H)} \circ F_t \circ i_\mu^{(H)} = F_t^\mu \circ \pi_\mu^{(H)}.$$

(v) F_t^μ is Hamiltonian on $M_\mu^{(H)}$ for the **reduced Hamiltonian** $h_\mu^{(H)} : M_\mu^{(H)} \to \mathbb{R}$, $h_\mu^{(H)} \circ \pi_\mu^{(H)} = h \circ i_\mu^{(H)}$. X_h and $X_{h_\mu^{(H)}}$ are $\pi_\mu^{(H)}$-related.

(vi) $h, k \in C^\infty(M)^G \Rightarrow \{h, k\} \in C^\infty(M)^G$ and

$$\{h, k\}_\mu^{(H)} = \{h_\mu^{(H)}, k_\mu^{(H)}\}_{M_\mu^{(H)}}$$

where $\{\cdot, \cdot\}_{M_\mu^{(H)}}$ is the Poisson bracket induced by the symplectic structure on $M_\mu^{(H)}$.

Sjamaar point reduction principle

GOAL: Realize the strata as usual reduced spaces

Recall: we start with a proper symplectic G-action on (M, ω)

- $m \in M$ is fixed, $H := G_m$, $\mu := \mathbf{J}(m)$.
- $N(H)^m := \{n \in N(H) \mid n \cdot M_H^m \subset M_H^m\}$.

$N(H)^m$ is open, hence closed, in $N(H)$. Also $H \subset N(H)^m$. Thus Lie$(N(H)^m/H) =$ Lie$(N(H)/H) =: \mathfrak{l}$

- $L^m := N(H)^m/H$ acts freely, properly, and symplectically on M_H^m with momentum map

$$\mathbf{J}_{L^m} : M_H^m \ni z \longmapsto \Lambda(\mathbf{J}|_{M_H^m}(z) - \mu) \in (\mathrm{Lie}(L^m))^*$$

- $\Lambda : (\mathfrak{g}_m^\circ)^H \to (\mathrm{Lie}(L^m))^*$, L^m-equivariant isomorphism

$$\left\langle \Lambda(\beta), \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)H \right\rangle = \langle \beta, \xi \rangle,$$

$\beta \in (\mathfrak{g}_m^\circ)^H$, $\xi \in$ Lie$(N(H)^m) =$ Lie$(N(H))$

- $\mathfrak{g}_m^\circ \subseteq \mathfrak{g}^*$ denotes the annihilator of $\mathfrak{g}_m$ in $\mathfrak{g}^*$
- $(\mathfrak{g}_m^\circ)^H$ are the H-fixed points in $\mathfrak{g}_m^\circ$
- Non-equivariance one-cocycle of $\mathbf{J}_{L^m}$

$$\tau : L^m \ni l \longmapsto \Lambda(c(n) + n \cdot \mu - \mu) \in (\mathrm{Lie}(L^m))^*$$

for $l = nH \in L^m$ and $n \in N(H)^m$.

(i) $\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \to M_\mu^{(H)} := [\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)]/G_\mu$ is a smooth fiber bundle with fiber G_μ/H and structure group $N_{G_\mu}(H)^m/H$.

(ii)

$$\begin{aligned}(M_H^m)_0 &:= \mathbf{J}_{L^m}^{-1}(0)/L_0^m \\ &= [\mathbf{J}^{-1}(\mu) \cap M_H^m]/(N_{G_\mu}(H)^m/H)\end{aligned}$$

$L_0^m \neq L^m$, in general (recall, the L^m-action is affine).

(iii) $\pi_0 : \mathbf{J}^{-1}_{L^m}(0) \to (M_H^m)_0$ is a principal L_0^m-bundle. G_μ/H is a right $(N_{G_\mu}(H)^m/H)$-space and $\mathbf{J}^{-1}(\mu)\cap M_H^m$ is a left $(N_{G_\mu}(H)^m/H)$-space. The associated bundle with fiber G_μ/H

$$G_\mu/H \times_{N_{G_\mu}(H)^m/H} \left(\mathbf{J}^{-1}(\mu) \cap M_H^m\right) \longrightarrow [\mathbf{J}^{-1}(\mu) \cap M_H^m]/(N_{G_\mu}(H)^m/H).$$

is G_μ-symplectomorphic to

$$\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \longrightarrow M_\mu^{(H)},$$

which means that

- $G_\mu/H \times_{N_{G_\mu}(H)^m/H} \left(\mathbf{J}^{-1}(\mu) \cap M_H^m\right) \overset{\sim}{\longleftrightarrow} \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)$ is a G_μ-diffeomorphism

- $(M_H^m)_0 = \mathbf{J}^{-1}_{L^m}(0)/L_0^m = (\mathbf{J}^{-1}(\mu) \cap M_H^m)/(N_{G_\mu}(H)^m/H)$ is symplectomorphic to $M_\mu^{(H)}$.

• $\{\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \mid \mathbf{J}(z) = \mu\}$ forms a Whitney (B) stratification of $\mathbf{J}^{-1}(\mu)$.

• $\{M_\mu^{(H)} \mid (H)\}$ is a symplectic Whitney (B) stratification of the cone space $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$.

• Each connected component of M_μ contains a unique open stratum that is connected, open, and dense in the connected component of M_μ that contains it.

There are similar theorems for orbit reduction. In the diagram, at every level, the corresponding spaces are isomorphic and in the respective category.

In the diagram below:
• L_μ is an isomorphism of cone (hence Whitney (B)) stratified spaces; in particular, L_μ it is a homeomorphism
• $L_\mu^{(H)}$ is the restriction of L_μ to the stratum determined by $H := G_m$
• $f_\mu^{(H)}$ and $f_{\mathcal{O}_\mu}^{(H)}$ are the Sjamaar principle symplectomorphisms

$$
\begin{array}{ccccl}
\mathbf{J}^{-1}(\mu) & \xrightarrow{\;l_\mu\;} & \mathbf{J}^{-1}(\mathcal{O}_\mu) & & \text{inclusion} \\
\Big\downarrow \pi_\mu & & \Big\downarrow \pi_{\mathcal{O}_\mu} & & \text{projections} \\
\mathbf{J}^{-1}(\mu)/G_\mu & \xrightarrow{\;L_\mu\;} & \mathbf{J}^{-1}(\mathcal{O}_\mu)/G & & \text{stratified isomorphism} \\
\Big\uparrow & & \Big\uparrow & & \text{stratum inclusions} \\
\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)/G_\mu & \xrightarrow{\;L_\mu^{(H)}\;} & G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)/G & & \text{symplectomorphism} \\
\Big\uparrow f_\mu^{(H)} & & \Big\uparrow f_{\mathcal{O}_\mu}^{(H)} & & \text{symplectomorphism} \\
\mathbf{J}_{L^m}^{-1}(0)/L_0^m & \xrightarrow{\;L_0\;} & \mathbf{J}_{L^m}^{-1}(\mathcal{O}_0)/L^m & & \text{symplectomorphism} \\
\| & & \| & & \\
\left[\mathbf{J}^{-1}(\mu) \cap M_H^m\right] / \left(N_{G_\mu}(H)^m/H\right) & & \left[\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M_H^m\right] / \left(N(H)^m/H\right) & &
\end{array}
$$

Cotangent bundle reduction – embedding

$\Phi : G \times Q \to Q$ left free proper action $\Longrightarrow Q_\mu := Q/G_\mu$ is a smooth manifold and $\pi_{Q,Q_\mu} : Q \to Q_\mu$ is a principal G_μ-bundle.

Lift Φ to a G-action on (T^*Q, ω_Q); it is free, proper, and it admits an equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \alpha_q(\xi_Q(q)), \quad \forall \alpha_q \in T^*Q, \ \xi \in \mathfrak{g}.$$

Reduce at $\mu \in \mathfrak{g}^*$ to get a symplectic manifold $((T^*Q)_\mu, \Omega_\mu)$.

HYPOTHESIS: $\exists \, \alpha_\mu \in \Omega^2(Q)$, G_μ-invariant, taking values in $\mathbf{J}^{-1}(\mu)$.

$\exists! \, \beta_\mu \in \Omega^2(Q_\mu)$ such that $\pi^*_{Q,Q_\mu} \beta_\mu = \mathbf{d}\alpha_\mu$.

β_μ is closed (not exact, in general).

Note: α_μ does not drop to Q_μ whereas $\mathbf{d}\alpha_\mu$ does.

Define $B_\mu := \pi^*_{Q_\mu} \beta_\mu \in \Omega^2(T^*Q)$, where $\pi_{Q_\mu} : T^*Q_\mu \to Q_\mu$ projection.

Cotangent Bundle Reduction – Embedding Version. There is a symplectic embedding

$$\varphi_\mu : ((T^*Q)_\mu, (\omega_Q)_\mu) \longrightarrow (T^*Q_\mu, \omega_{Q_\mu} - B_\mu),$$

onto the vector subbundle $[T\pi_{Q,G_\mu}(V)]^\circ \subseteq T^*Q_\mu$, where $V \subset TQ$ is the vector subbundle consisting of vectors tangent to the G-orbits, i.e., its fiber at $q \in Q$ equals $V_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$, and $^\circ$ denotes the annihilator for the natural duality pairing between TQ_μ and T^*Q_μ.

$\varphi_\mu : ((T^*Q)_\mu, (\omega_Q)_\mu) \xrightarrow{\sim} (T^*Q_\mu, \omega_{\mathrm{can}} - B_\mu)$ symplectic $\Longleftrightarrow \mathfrak{g} = \mathfrak{g}_\mu$.

Let $\mathcal{A} \in \Omega^1(Q;\mathfrak{g})$ be a principal connection on the G-principal bundle $\pi_{Q,Q/G} : Q \to Q/G$ and $\mathcal{B} \in \Omega^2(Q;\mathfrak{g})$ its curvature.

Can choose $\alpha_\mu(q) := \mathcal{A}(q)^*\mu \Longrightarrow \mathbf{d}\alpha_\mu = \left\langle \mu, \mathcal{B} + \frac{1}{2}[\mathcal{A} \overset{\wedge}{,} \mathcal{A}] \right\rangle \in \Omega^2(Q)$.

Recall: $B_\mu = \pi^*_{Q_\mu}\beta_\mu \in \Omega^2(T^*Q_\mu)$, $\beta_\mu = \pi^*_{Q,Q_\mu}\mathbf{d}\alpha_\mu \in \Omega^2(Q_\mu)$.

Cotangent bundle reduction – fibration

$\Phi : G \times Q \to Q$ left free proper action

Cotangent Bundle Reduction—Bundle Version Reduced space $(T^*Q)_\mu \to T^*(Q/G)$ is a locally trivial fiber bundle, typical fiber $\mathcal{O}_\mu$.

This is not good enough because it does not say anything about the symplectic form on $(T^*Q)_\mu$ in terms of the symplectic structure of $T^*(Q/G)$ and the orbit symplectic structure on $\mathcal{O}_\mu$.

Need to study first the Poisson situation to fix the setup, also easier.

Let $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$ be a principal connection on $\pi_{Q,Q/G} : Q \to Q/G$.

$H_q = \{v_q \in T_qQ \mid \mathcal{A}(v_q) = 0\}$ **horizontal space** at $q \in Q$
$V_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$ **vertical space** at $q \in Q$

$T_qQ \ni v_q \longmapsto \mathrm{ver}_q(v_q) := [\mathcal{A}(q)(v_q)]_Q(q) \in V_q$ **vertical projection**
$T_qQ \ni v_q \longmapsto \mathrm{hor}_q(v_q) := v_q - \mathrm{ver}_q(v_q)$ **horizontal projection**

$T_q\pi_{Q,Q/G}|H_q : H_q \to T_{[q]}(Q/G)$ isomorphism with inverse
$\mathrm{Hor}_q := \left[T_q\pi_{Q,Q/G}|_{H_q}\right]^{-1} : T_{[q]}(Q/G) \to H_q$, **horizontal lift** at $q \in Q$

$\pi_{Q\times\mathfrak{g},Q/G} : \tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G \to Q/G$, the **adjoint bundle**; a vector bundle with fibers isomorphic to $\mathfrak{g}$; $\pi_{Q\times\mathfrak{g},\tilde{\mathfrak{g}}} : Q \times \mathfrak{g} \to \tilde{\mathfrak{g}}$ projection

Vector bundle isomorphism

$$\alpha_{\mathcal{A}} : TQ/G \ni [v_q] \longmapsto (T_q\pi(v_q), [q, \mathcal{A}(q)(v_q)]) \in T(Q/G) \oplus \tilde{\mathfrak{g}}$$

with inverse

$$\alpha_{\mathcal{A}}^{-1} : T(Q/G) \oplus \tilde{\mathfrak{g}} \ni \left(v_{[q]}, [q, \xi]\right) \longmapsto [\mathrm{Hor}_q v_{[q]} + \xi_Q(q)] \in TQ/G$$

$$(\alpha_{\mathcal{A}}^{-1})^* : T^*Q/G \ni [\alpha_q] \longmapsto \left(\mathrm{Hor}_q^* \alpha_q, [q, \mathbf{J}(\alpha_q)]\right) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

where $\mathrm{Hor}_q^* : T_q^*Q \to T_{[q]}^*(Q/G)$ is dual to $\mathrm{Hor}_q : T_{[q]}(Q/G) \to T_qQ$.

$\mathcal{A} \in \Omega^1(Q;\mathfrak{g})$ induces an affine connection on $T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^* \to Q/G$.

For $f \in C^\infty(T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*)$, $w = (\alpha_{[q]}, [q,\mu]) \in W := T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*$, $v_{\alpha_{[q]}} \in T_{\alpha_{[q]}}(T^*(Q/G))$, the **exterior covariant derivative** is:

$\mathbf{d}_{\widetilde{\mathcal{A}}} f(w) \in T_{\alpha_{[q]}}(T^*(Q/G))$, $\qquad \pi_{Q/G} : T^*(Q/G) \to Q/G$

$$\mathbf{d}_{\widetilde{\mathcal{A}}} f(w)\left(v_{\alpha_{[q]}}\right) := \mathbf{d}f(w)\left(v_{\alpha_{[q]}}, T_{(q,\mu)}\pi_{Q\times\mathfrak{g},\widetilde{\mathfrak{g}}}\left(\mathrm{Hor}_q\left(T_{\alpha_{[q]}}\pi_{Q/G}\left(v_{\alpha_{[q]}}\right)\right), 0\right)\right)$$

Push forward by $(\alpha_{\mathcal{A}}^{-1})^*$ Poisson bracket. If $f,g \in C^\infty\left(T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*\right)$

$$\begin{aligned}\{f,g\}_W(w) &= \omega_{Q/G}\left(\alpha_{[q]}\right)\left(\mathbf{d}_{\widetilde{\mathcal{A}}} f(w)^\sharp, \mathbf{d}_{\widetilde{\mathcal{A}}} g(w)^\sharp\right) \\ &\quad + \left\langle [q,\mu], \widetilde{\mathcal{B}}\left(\alpha_{[q]}\right)\left(\mathbf{d}_{\widetilde{\mathcal{A}}} f(w)^\sharp, \mathbf{d}_{\widetilde{\mathcal{A}}} g(w)^\sharp\right)\right\rangle - \left\langle w, \left[\frac{\delta f}{\delta w}, \frac{\delta g}{\delta w}\right]\right\rangle\end{aligned}$$

$\frac{\delta f}{\delta w} \in (T(Q/G) \oplus \widetilde{\mathfrak{g}})_{\alpha_{[q]}}$ is the **fiber derivative**

$$\left\langle w', \frac{\delta f}{\delta w}\right\rangle := \left.\frac{d}{dt}\right|_{t=0} f(w + tw'), \quad w, w' \in (T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*)_{\alpha_{[q]}}$$

$$\widetilde{\mathcal{B}} := \pi_{Q/G}^*\mathcal{B} \in \Omega^2(T^*(Q/G);\widetilde{\mathfrak{g}}), \quad \mathcal{B} \in \Omega^2(T^*(Q/G);\widetilde{\mathfrak{g}})$$

$$\mathcal{B}([q])\left(T_q\pi_{Q/G}u_q, T_q\pi_{Q/G}v_q\right) := [q, \mathrm{Curv}_{\mathcal{A}}(q)(u_q,v_q)]$$

Determine the symplectic leaves of the gauged Lie-Poisson bracket on $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$? Solved by Perlmutter in his 1999 thesis and in final form Marsden-Perlmutter [2000].

$\mathcal{O} \subset \mathfrak{g}^*$ codajoint orbit

$\widetilde{\mathcal{O}} := (Q \times \mathcal{O})/G \to Q/G$ associated fiber bundle. $\pi_{Q\times\mathcal{O},\widetilde{\mathcal{O}}} : Q \times \mathcal{O} \to \widetilde{\mathcal{O}}$

$T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}} := \left\{ \left(\alpha_{[q]}, [q, \nu] \right) \mid q \in Q,\, \nu \in \mathcal{O},\, \alpha_{[q]} \in T_{\alpha_{[q]}}(T^*Q) \right\}$ is a fiber bundle over Q/G whose fiber at $[q] \in Q/G$ is $T^*_{[q]}(Q/G) \times \widetilde{\mathcal{O}}_{[q]}$

$$(\alpha_{\mathcal{A}}^{-1})^* \left(\mathbf{J}^{-1}(\mathcal{O}/G) \right) = T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}} \subset T^*(Q/G) \oplus \tilde{\mathfrak{g}}$$

So, reduced symplectic form $\omega_{\mathcal{O}}$ on $(T^*Q)_{\mathcal{O}} := \mathbf{J}^{-1}(\mathcal{O})/G$ pushes forward by $(\alpha_{\mathcal{A}}^{-1})^*$ to a symplectic form $\omega_{\mathcal{A}}$ on $T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}}$:

$$\omega_{\mathcal{A}} = \omega_{Q/G} - \beta$$

where $\beta \in \Omega^2\left(\widetilde{\mathcal{O}}\right)$ is uniquely determined by

$$\pi^*_{Q\times\mathcal{O},\widetilde{\mathcal{O}}}\beta = \mathbf{d}\alpha + \pi^*_{Q\times\mathcal{O},\mathcal{O}}\,\omega^+_{\mathcal{O}}, \qquad \alpha \in \Omega^1(Q\times\mathcal{O}),$$

$$\alpha(q,\nu)\left(u_q, -\mathrm{ad}^*_{\xi'}\nu\right) := \langle \nu, \mathcal{A}(q)(u_q)\rangle, \quad q\in Q,\ u_q \in T_qQ,\ \nu\in\mathcal{O},\ \xi'\in\mathfrak{g}.$$

$\mathbf{d}\alpha$ has the *explicit* expression

$$\mathbf{d}\alpha(q,\nu)\left(\left(u_q, -\mathrm{ad}^*_{\xi'}\nu\right), \left(v_q, -\mathrm{ad}^*_{\eta'}\nu\right)\right) = \left\langle \nu,\ [\eta',\xi] + [\eta,\xi'] + [\xi,\eta] + \mathrm{Curv}_{\mathcal{A}}(q)(u_q,v_q)\right\rangle$$

$q\in Q$, $\nu\in\mathcal{O}$, $\xi,\xi',\eta,\eta'\in\mathfrak{g}$, $u_q, v_q \in T_qQ$, where

$$u_q = \xi_Q(q) + \mathrm{hor}_q\, u_q, \qquad v_q = \eta_Q(q) + \mathrm{hor}_q\, v_q$$

is the vertical-horizontal splitting on T_qQ given by $\mathcal{A}$.

So, $\left((T^*Q)_{\mathcal{O}}, (\omega_Q)_{\mathcal{O}}\right) \overset{\sim}{\longleftrightarrow} \left(T^*(Q/G)\times_{Q/G}\widetilde{\mathcal{O}},\ \omega_{\mathcal{A}}\right)$ symplectomorphism

Reconstruction of dynamics

Given: Integral curve $c_\mu(t)$ of $X_{h_\mu} \in \mathfrak{X}\left((T^*Q)_\mu\right)$. Let $\alpha_q \in \mathbf{J}^{-1}(\mu)$. Find integral curve $c(t)$ of $X_h \in \mathfrak{X}(T^*Q)$ with initial condition α_q.

Solution: $c(t) = g(t) \cdot d(t)$. Let $\mathcal{A} \in \Omega^1(\mathbf{J}^{-1}(\mu); \mathfrak{g}_\mu)$ be a connection and take $d(t)$ to be the horizontal lift through α_q of $c_\mu(t)$. Solve $\dot{g}(t) = T_e L_{g(t)} \xi(t)$, $g(0) = e$. So, it all comes down to:

- Choice of a convenient connection $\mathcal{A} \in \Omega^1(\mathbf{J}^{-1}(\mu); \mathfrak{g}_\mu)$.
- Finding $\xi(t) \subset \mathfrak{g}_\mu$ in terms of $d(t)$.

1.) $G_\mu = S^1$ or $\mathbb{R}$. Let $\zeta \in \mathfrak{g}_\mu$ be a basis. Identify $\mathbb{R} \ni a \stackrel{\sim}{\longleftrightarrow} a\zeta \in \mathfrak{g}_\mu$. Connection $\mathcal{A} = \frac{1}{\langle \mu, \zeta \rangle} \theta_\mu \in \Omega^1(\mathbf{J}^{-1}(\mu))$, where θ_μ is the pull back to $\mathbf{J}^{-1}(\mu)$ of the canonical $\theta_Q \in \Omega^1(T^*Q)$; $\omega_Q = -\mathbf{d}\theta_Q$ canonical symplectic form on T^*Q. The curvature is $\operatorname{Curv}_{\mathcal{A}} = -\frac{1}{\langle \mu, \zeta \rangle} \omega_\mu \in \Omega^2((T^*Q)_\mu)$. Then $\xi(t) = \mathbf{d}h(\Lambda)(d(t))$, where $\Lambda = p_i \frac{\partial}{\partial p_i}$ (unique vector field on T^*Q satisfying $\mathbf{d}\theta_Q(\Lambda, \cdot) = \theta_Q$).

2.) Let $\mathfrak{A} \in \Omega^1(Q; \mathfrak{g}_\mu)$ be a connection on the left G_μ-principal bundle $Q \to Q/G_\mu$. $\mathfrak{A}$ induces a connection $\mathcal{A} \in \Omega^1(\mathbf{J}^{-1}(\mu); \mathfrak{g}_\mu)$ by

$$\mathcal{A}(\alpha_q)\left(V_{\alpha_q}\right) := \mathfrak{A}(q)\left(T_{\alpha_q}\pi_Q\left(V_{\alpha_q}\right)\right), \quad q \in Q,\, \alpha_q \in T_q^*Q,\, V_{\alpha_q} \in T_{\alpha_q}(T^*Q).$$

Then $\xi(t) = \mathfrak{A}(q(t))\Big(\mathbb{F}h(d(t)\Big) \subset \mathfrak{g}_\mu$, $\mathbb{F}h : T^*Q \to TQ$ fiber derivative, $q(t) := \pi_Q(d(t)) \subset Q$.

3.) Let $(Q, \langle\langle\cdot,\cdot\rangle\rangle)$ be a Riemannian manifold and G act by isometries. The **mechanical connection** is defined by requiring that its horizontal bundle is the orthogonal to the vertical bundle.

$$\mathfrak{A}_{\mathsf{mech}}(q)(u_q) := \mathbb{I}_\mu(q)^{-1}\mathbf{J}\left(u_q^\flat\right), \quad q \in Q,\, u_q \in T_qQ$$

$u_q^\flat := \langle\langle u_q, \cdot\rangle\rangle \in T_q^*Q$, $\mathbb{I}_\mu(q) : \mathfrak{g}_\mu \xrightarrow{\sim} \mathfrak{g}_\mu^*$ is the **μ-locked inertia tensor** defined for each $q \in Q$ by $\mathbb{I}_\mu(q)(\zeta)(\eta) := \left\langle\left\langle \zeta_Q(q), \eta_Q(q)\right\rangle\right\rangle$. Special situation of **2.)**. Then

$$\xi(t) = \mathfrak{A}_{\mathsf{mech}}(q(t))\left(d(t)^\sharp\right).$$

4.) **Simple mechanical systems.** The Hamiltonian is of the form $h = k + v \circ \pi_Q$, where k is the kinetic energy of the cometric on T^*Q determined by a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on Q and $v \in C^\infty(Q)$. G acts by isometries on Q and the potential energy v is G-invariant. The reconstruction method is quite explicit in this case.

Given is $\alpha_q \in \mathbf{J}^{-1}(\mu) \subset T_q^*Q$ and the solution $c_\mu(t) \subset (T^*Q)_\mu$ of X_{h_μ} with initial condition $[\alpha_q] \in (T^*Q)_\mu$.

Step 1.) $\varphi_\mu : \left((T^*Q)_\mu, (\omega_Q)_\mu \right) \hookrightarrow \left(T^*(Q/G_\mu), \omega_{Q/G_\mu} - B_\mu \right)$, symplectic embedding onto a vector subbundle, B_μ induced by the mechanical connection. Then $\varphi_\mu(c_\mu(t))$ is an integral curve of the Hamiltonian system on $\left(T^*(Q/G_\mu), \omega_{Q/G_\mu} - B_\mu \right)$ given by the kinetic energy of the quotient Riemannian metric on Q/G_μ and the quotient of the amended potential $v_\mu := h \circ \alpha_\mu \in C^\infty(Q)$. Compute the curves
$\varphi_\mu(c_\mu(t)) \subset T^*(Q/G_\mu)$ and
$q_\mu(t) := \pi_{Q/G_\mu}(\varphi_\mu(c_\mu(t))) \subset Q/G_\mu$.

Step 2.) Using the mechanical connection $\mathfrak{A}_{\text{mech}} \in \Omega^1(Q;\mathfrak{g}_\mu)$, horizontally lift $q_\mu(t) \in Q/G\mu$ to a curve $q_h(t) \subset Q$ with $q_h(0) = q$.

Step 3.) Determine $\xi(t) \subset \mathfrak{g}_\mu$ from the algebraic equation

$$\left\langle\!\!\left\langle \xi(t)_Q(q_h(t)), \eta_Q(q_h(t)) \right\rangle\!\!\right\rangle = \langle \mu, \eta \rangle\,, \ \forall \eta \in \mathfrak{g}_\mu.$$

So, $\dot{q}_h(0)$ and $\xi(0)_Q(q)$ are the horizontal and vertical components of the vector $\alpha_q^\sharp \in T_qQ$.

Step 4.) Solve $\dot{g}(t) = T_eL_{g(t)}\xi(t)$ in G_μ with $g(0) = e$.

Step 5.) With $q_h(t)$ from **Step 2.)** and $g(t)$ from **Step 4.)**, define $q(t) := g(t) \cdot q_h(t)$. This is the base integral curve of the simple mechanical system with Hamiltonian $h = k + v \circ \pi_Q$ satisfying $q(0) = q$. The curve $\dot{q}(t)^\flat \subset T^*Q$ is the integral curve of X_h with $\dot{q}(t)^\flat(0) = \alpha_q$.

Interesting special cases

(a) If G_μ is Abelian, equation in **Step 4.)** has the solution

$$g(t) = \int_0^t \xi(s) ds.$$

(b) $G_\mu = S^1$, ζ basis of $\mathfrak{g}_\mu$. Can solve for $\xi(t)$ in **Step 3.)**, namely

$$\xi(t) = \frac{\langle \mu, \zeta \rangle}{\|\zeta_Q(q_h(t))\|^2} \zeta$$

and hence

$$q(t) = \exp\left(\langle \mu, \zeta \rangle \int_0^t \frac{ds}{\|\zeta_Q(q_h(s))\|^2} \right) \cdot q_h(t)$$

(c) If G is compact and $(\cdot, \cdot)$ is a positive definite metric, invariant under the adoint G-action on $\mathfrak{g}$, and satisfying

$$(\zeta, \eta) = \left\langle\!\left\langle \zeta_Q(q), \eta_Q(q) \right\rangle\!\right\rangle, \quad \forall q \in Q, \ \zeta, \eta \in \mathfrak{g},$$

then $\xi \in \mathfrak{g}_\mu$ is uniquely determined by $(\xi, \cdot) = \mu|_{\mathfrak{g}_\mu}$ and $g(t) = \exp(t\xi)$.

(d) If G is solvable, let $\{\xi_1, \ldots, \xi_n\} \subset \mathfrak{g}$ be a basis. Write

$$g(t) = \exp(f_1(t)\xi_1) \cdots \exp(f_n(t)\xi_n).$$

Wei and Norman [1964] have shown that $\dot{g}(t) = T_e L_{g(t)} \xi(t)$ can be solved by quadratures for the all the functions $f_1(t), \ldots, f_n(t)$.

(e) If $\dot{\xi}(t) = \alpha(t)\xi(t)$ for a known function $\alpha(t)$, then $g(t) = \exp(f(t)\xi(t))$ solves $\dot{g}(t) = T_e L_{g(t)} \xi(t)$, where

$$f(t) = \int_0^t \exp\left(\int_t^s \alpha(r)dr\right) ds.$$

The conditions in **(c)** are very strong, but they hold for the Kaluza-Klein construction. Many of these formulas are very useful when one wants to compute geometric phases.

What happens if the action of G on Q is not free? Only partial results of Perlmutter and Rodríguez-Olmos. General case is open.