Stochastic Euler-Poincaré reduction.

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References

- **Arnaudon, Marc; Chen, Xin; Cruzeiro, Ana Bela;** *Stochastic Euler-Poincaré reduction.* J. Math. Phys. 55 (2014), no. 8, 17pp
- **Chen, Xin; Cruzeiro, Ana Bela; Ratiu, Tudor S.;** *Constrained and stochastic variational principles for dissipative equations with advected quantities.* arXiv:1506.05024

1. Deterministic framework
   - Euler-Poincaré equations
   - Diffeomorphism group on a compact Riemannian manifold
   - Volume preserving diffeomorphism group
   - Lagrangian paths
   - Characterization of the geodesics on $\left(G_V^s, \langle\cdot,\cdot\rangle^0\right)$
   - Euler-Poincaré equation on G_V^s

2. Stochastic framework
   - Semi-martingales in a Lie group
   - Stochastic Euler-Poincaré reduction
   - Group of volume preserving diffeomorphisms
   - Navier-Stokes and Camassa-Holm equations

- Let M be a Riemannian manifold and $L : TM \times [0, T] \to \mathbb{R}$ a Lagrangian on M.
- Let $q \in C^1_{a,b}([0,T]; M) := \{q \in C^1([0,T], M),\ q(0) = a,\ q(T) = b\}$.
- The action functional $\mathscr{C} : C^1_{a,b}([0,T]; M) \to \mathbb{R}$ is defined by

$$\mathscr{C}(q(\cdot)) := \int_0^T L(q(t), \dot{q}(t), t)\, dt.$$

- The critical points for $\mathscr{C}$ satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

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- Suppose that the configuration space $M = G$ is a Lie group and $L : TG \to \mathbb{R}$ is a left invariant Lagrangian:

$$\ell(\xi) := L(e, \xi) = L(g, g \cdot \xi), \ \forall \xi \in T_eG, \ g \in G.$$

(here and in the sequel, $g \cdot \xi = T_eL_g\xi$)

- The action functional $\mathscr{C} : C^1_{a,b}([0, T]; G) \to \mathbb{R}$ is defined by

$$\mathscr{C}(g(\cdot)) := \int_0^T L\left(g(t), \dot{g}(t)\right) dt = \int_0^T \ell(\xi(t))\, dt,$$

where $\xi(t) := g(t)^{-1} \cdot \dot{g}(t)$.

- **[J.E. Marsden, T. Ratiu 1994] [J.E. Marsden, J. Scheurle 1993]**: $g(\cdot)$ is a critical point for $\mathscr{C}$ if and only if it satisfies the Euler-Poincaré equation on T_e^*G

$$\frac{d}{dt}\left(\frac{d\ell}{d\xi}\right) - \mathrm{ad}^*_{\xi(t)}\left(\frac{d\ell}{d\xi}\right) = 0,$$

where $\mathrm{ad}^*_\xi : T_e^*G \to T_e^*G$ is the dual action of $\mathrm{ad}_\xi : T_eG \to T_eG$:

$$\langle \mathrm{ad}^*_\xi \eta, \theta\rangle = \langle \eta, \mathrm{ad}_\xi \theta\rangle, \quad \eta \in T_e^*G, \quad \theta \in T_eG.$$

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- We will be interested in variations $\xi(\cdot)$ satisfying

$$\dot\xi(t) = \dot\nu(t) + \mathrm{ad}_{\xi(t)}\,\nu(t) \quad \text{for some} \quad \nu \in C^1([0,T],\ T_eG),$$

which is equivalent to the variation of $g(\cdot)$ with the perturbation $g^\varepsilon(t) = g(t)e_{\varepsilon,\nu}(t)$, where $e_{\varepsilon,\nu}(t)$ is the unique solution to the following ODE on G:

$$\begin{cases} \frac{d}{dt}e_{\varepsilon,\nu}(t) = & \varepsilon e_{\varepsilon,\nu}(t)\cdot\dot\nu(t), \\ e_{\varepsilon,\nu}(0) = & e. \end{cases}$$

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- Let M be a n-dimensional compact Riemannian manifold. We define

$$G^s := \left\{ g : M \to M \text{ a bijection }, g, g^{-1} \in H^s(M,M) \right\},$$

where $H^s(M,M)$ denotes the manifold of Sobolev maps of class $s > 1 + \frac{n}{2}$ from M to itself.
- If $s > 1 + \frac{n}{2}$ then G^s is a C^∞ Hilbert manifold.
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- The tangent space $T_\eta G^s$ at arbitrary $\eta \in G^s$ is

$$T_\eta G^s = \left\{U : M \to TM \text{ of class } H^s,\ U(m) \in T_{\eta(m)}M\right\}.$$

- The Riemannian structure on M induces the weak L^2, or hydrodynamic, metric $\langle\cdot,\cdot\rangle^0$ on G^s given by

$$\langle U, V\rangle_\eta^0 := \int_M \langle U_\eta(m), V_\eta(m)\rangle_m \, d\mu_g(m),$$

for any $\eta \in G^s$, $U, V \in T_\eta G^s$. Here $U_\eta := U \circ \eta^{-1} \in T_e G^s$ and μ_g denotes the Riemannian volume asociated with (M, g).

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- Let ∇ be the Levi-Civita connection associated with the Riemannian manifold (M,g). We define a right invariant connection ∇^0 on G^s by

$$\left(\nabla^0_{\tilde X}\tilde Y\right)(\eta) := \frac{\partial}{\partial t}\Big|_{t=0}\left(\tilde Y(\eta_t)\circ\eta_t^{-1}\right)\circ\eta + \left(\nabla_{X_\eta}Y_\eta\right)\circ\eta,$$

where $\tilde X, \tilde Y \in \mathscr{L}(G^s)$, $X_\eta := \tilde X\circ\eta^{-1}$, $Y_\eta := \tilde Y\circ\eta^{-1} \in \mathscr{L}^s(M)$, and η is a C^1 curve in G^s such that $\eta_0 = \eta$ and $\frac{d}{dt}\Big|_{t=0}\eta_t = \tilde X(\eta)$. Here $\mathscr{L}(G^s)$ denotes the set of smooth vector fields on G^s.
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- For $s > 1 + \frac{n}{2}$, let

$$G_V^s := \{g, g \in G^s,\ g \text{ is volume preserving}\}.$$

- G_V^s is still a topological group.
- The tangent space $T_e G_V^s$ is

$$\mathscr{G}_V^s = T_e G_V^s = \{U,\ U \in T_e G^s,\ \mathrm{div}(U) = 0\}.$$

- The L^2-metric $\langle\cdot,\cdot\rangle^0$ and its Levi-Civita connection $\nabla^{0,V}$ are defined on G_V^s by orthogonal projection. More precisely the Levi Civita connection on G_V^s is given by $\nabla_X^{0,V} Y = P_e(\nabla_X^0 Y)$ with P_e the orthogonal projection on $\mathscr{G}_V^s$:

$$H^s(TM) = \mathscr{G}_V^s \oplus dH^{s+1}(M).$$

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- Consider the ODE on M

$$\begin{cases} \frac{d}{dt}\left(g_t(x)\right) & = u\left(t, g_t(x)\right) \\ g_0(x) & = x. \end{cases}$$

  Here $u(t,\cdot) \in T_eG^s$ for every $t > 0$.
- For every fixed $t > 0$, $g_t(\cdot) \in G^s(M)$. So $g \in C^1([0,T], G^s)$.
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$$\text{(E)} \quad \begin{cases} \frac{\partial u}{\partial t} & = -\nabla_u u - \nabla p \\ \operatorname{div} u & = 0 \end{cases}$$

- Notice that the term ∇p corresponds to the use of ∇^0 instead of $\nabla^{0,V}$: the first system rewrites as

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- If we take $\ell : T_e G_V^s \to \mathbb{R}$ as

$$\ell(X) := \langle X, X\rangle, \quad X \in T_e G_V^s,$$

and define the action functional $\mathscr{C} : C^1_{e,e}([0,T], G_V^s) \to \mathbb{R}$ by

$$\mathscr{C}(g(\cdot)) := \int_0^T \ell\left(\dot{g}(t)\cdot g(t)^{-1}\right)\, dt,$$

then a Lagrangian path $g \in C^2([0,T], G_V^s)$ integral path of u is a critical point of $\mathscr{C}$ if and only if u satisfies the Euler equation (E). **[J.E. Marsden, T. Ratiu 1994] [J.E. Marsden, J. Scheurle 1993]**

- **[S. Shkoller 1998]** If we take $\ell : T_eG_V^s \to \mathbb{R}$ as the H^1 metric

$$\ell(X) := \int_M \langle X, X\rangle_m \, d\mu_g(m) + \alpha^2 \int_M \langle \nabla X, \nabla X\rangle_m \, d\mu_g(m), \quad X \in T_eG_V^s,$$

and define the action functional $\mathscr{C} : C^1_{e,e}([0,T], G_V^s) \to \mathbb{R}$ in the same way as before, then a Lagrangian path $g \in C^2([0,T], G_V^s)$ integral path of u is a critical point of $\mathscr{C}$ if and only if u satisfies the Camassa-Holm equation

$$\begin{cases} \frac{\partial \nu}{\partial t} + u \cdot \nu + \alpha^2 \left(\nabla u\right)^* \cdot \Delta \nu & = \nabla p, \\ \nu & = (1+\alpha^2\Delta)u, \\ \mathrm{div}(u) & = 0. \end{cases}$$

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An $\mathbb{R}^n$-valued semimartingale ξ_t has a decomposition

$$\xi_t(\omega) = N_t(\omega) + A_t(\omega)$$

where (N_t) is a local martingale and (A_t) has finite variation.
If (N_t) is a martingale, then

$$\mathbb{E}[N_t|\mathscr{F}_s] = N_s, \quad t \geq s.$$

We are interested in semimartingales which furthermore satisfy

$$A_t(\omega) = \int_0^t a_s(\omega)\, ds.$$

Defining

$$\frac{D\xi_t}{dt} := \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{\xi_{t+\varepsilon} - \xi_t}{\varepsilon} \Big| \mathscr{F}_t\right],$$

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Itô formula :

$$f(\xi_t) = f(\xi_0) + \int_0^t \langle df(\xi_s), dN_s\rangle + \int_0^t \langle df(\xi_s), dA_s\rangle + \frac{1}{2}\int_0^t \mathrm{Hess} f(d\xi_s \otimes d\xi_s).$$

From this we see that ξ_t is a local martingale if and only if for all $f \in C^2(\mathbb{R}^n)$,

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This property becomes a definition for manifold-valued martingales.

Definition

Let $a_t \in T_{\xi_t}M$ an adapted process. If for all $f \in C^2(M)$

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Let $\mathscr{G} := T_eG$ be the Lie algebra of G.
Consider a countable family H_i, $i \geq 1$, of elements of $\mathscr{G}$, and $u \in C^1([0,T],\mathscr{G})$.
Consider the Stratonovich equation

$$\begin{aligned} dg_t &= \left(\sum_{i\geq 1} H_i \circ dW_t^i - \frac{1}{2}\nabla_{H_i}H_i\, dt + u(t)\, dt\right)\cdot g_t \\ g_0 &= e \end{aligned}$$

where the (W_t^i) are independent real valued Brownian motions. Itô formula writes

$$\begin{aligned} f(g_t) =& f(g_0) + \sum_{i\geq 1}\int_0^t \langle df(g_s), H_i dW_s^i\rangle + \int_0^t \langle df(g_s), u(s)g_s\, ds\rangle \\ &+ \frac{1}{2}\sum_{i\geq 1}\int_0^t \mathrm{Hess} f(H_i(g_s), H_i(g_s))\, ds. \end{aligned}$$

This implies that $\dfrac{Dg_t}{dt} = u(t)g_t$.

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If (H_i) is an orthonormal basis, $\nabla_{H_i}H_i = 0$, ∇ is the Levi Civita connection associated to the metric and $u \equiv 0$, then g_t is a Brownian motion in G.

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On the space $\mathscr{S}(G)$ of G-valued semimartingales define

$$J(\xi) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\|\frac{D\xi}{dt}\right\|^2 dt\right].$$

Perturbation: for $v \in C^1([0,T],\mathscr{G})$ satisfying $v(0) = v(T) = 0$ and $\varepsilon > 0$, let $e_{\varepsilon,v}(\cdot) \in C^1([0,T],G)$ the flow generated by εv:

$$\begin{cases} \frac{d}{dt}e_{\varepsilon,v}(t) & = \varepsilon\dot{v}(t)\cdot e_{\varepsilon,v}(t) \\ e_{\varepsilon,v}(0) & = e \end{cases}$$

Definition

We say that $g \in \mathscr{S}(G)$ is a critical point of J if for all $v \in C^1([0,T],\mathscr{G})$ satisfying $v(0) = v(T) = 0$,

$$\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} g_{\varepsilon,v} = 0 \quad \text{where} \quad g_{\varepsilon,v}(t) = e_{\varepsilon,v}(t)g(t).$$

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Theorem

g is a critical point of J if and only if

$$\frac{du(t)}{dt} = -\mathrm{ad}^*_{\tilde{u}(t)} u(t) - K(u(t))$$

with

$$\tilde{u}(t) = u(t) - \frac{1}{2}\sum_{i\geq 1} \nabla_{H_i} H_i, \quad \langle \mathrm{ad}^*_u v, w\rangle = \langle v, \mathrm{ad}_u v\rangle$$

and $K : \mathscr{G} \to \mathscr{G}$ satisfies

$$\langle K(u), v\rangle = -\left\langle u, \frac{1}{2}\sum_{i\geq 1} \nabla_{\mathrm{ad}_v H_i} H_i + \nabla_{H_i}\left(\mathrm{ad}_v(H_i)\right)\right\rangle$$

Remark 1

If for all $i \geq 1$, $H_i = 0$, or $\nabla_u v = 0$ for all $u, v \in \mathscr{G}$, then $K(u) = 0$ and we get the standard Euler-Poincaré equation.

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Proposition

If for all $i \geq 1$, $\nabla_{H_i} H_i = 0$ then

$$K(u) = -\frac{1}{2} \sum_{i \geq 1} \nabla_{H_i} \cdot \nabla_{H_i} u + R(u, H_i) H_i.$$

In particular if (H_i) is an o.n.b. of $\mathscr{G}$ then

$$K(u) = -\frac{1}{2} \square u = -\frac{1}{2} \Delta u + \frac{1}{2} \mathrm{Ric}^\sharp u \quad \text{the Hodge Laplacian.}$$

Let

$$G_V^s = \{g : M \to M \quad \text{volume preserving bijection, such that} \quad g,\ g^{-1} \in H^s\}.$$

Assume $s > 1 + \frac{\dim M}{2}$. Then G_V^s is a C^∞ smooth manifold. Lie algebra

$$\mathscr{G}_V^s = T_e G_V^s = \{X : H^s(M, TM), \pi(X) = e, \operatorname{div}(X) = 0\}.$$

Notice that $\pi(X) = e$ means that X is a vector field on M: $X(x) \in T_x M$. On $\mathscr{G}_V^s$ consider the two scalar products

$$\langle X, Y\rangle^0 = \int_M \langle X(x), Y(x)\rangle\, dx$$

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The Levi Civita connection on G_V^s is given by $\nabla_X^{0V} Y = P_e(\nabla_X^0 Y)$ with ∇^0 the Levi Civita connection of $\langle\cdot,\cdot\rangle^0$ on G^s and P_e the orthogonal projection on $\mathscr{G}_V^s$:

$$H^s(TM) = \mathscr{G}_V^s \oplus dH^{s+1}(M).$$

One can find $(H_i)_{i\geq 1}$ such that for all $i \geq 1$, $\nabla_{H_i} H_i = 0$, $\operatorname{div}(H_i) = 0$, and

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Corollary

(1) g is a critical point of $J^{\langle\cdot,\cdot\rangle^0}$ if and only if u solves Navier-Stokes equation

$$\begin{cases} \frac{\partial u}{\partial t} & = -\nabla_u u + \frac{\nu}{2}\Delta u - \nabla p \\ \mathrm{div} u & = 0 \end{cases}$$

(2) Assume $M = \mathbb{T}^2$ the 2-dimensional torus. Then g is a critical point of $J^{\langle\cdot,\cdot\rangle^1}$ if and only if u solves Camassa-Holm equation

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